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Citation: Journal of Mathematical Physics **50**, 033501 (2009); doi: 10.1063/1.3081386
View online: http://dx.doi.org/10.1063/1.3081386
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/50/3?ver=pdfcov
Published by the AIP Publishing

Jump transformations and an embedding of $\mathcal{O}_\infty$ into $\mathcal{O}_2$

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(Received 28 September 2008; accepted 8 January 2009; published online 3 March 2009)

A measurable map F on a measure space induces a representation Π_F of a Cuntz algebra $\mathcal{O}_N$ when F satisfies a certain condition. For such two maps F and G and representations Π_F and Π_G associated with them, we show that Π_G is the restriction of Π_F when G is a jump transformation of F. Especially, the Gauss map τ_1 and the Farey map σ_1 induce representations Π_{τ_1} of $\mathcal{O}_\infty$ and that Π_{σ_1} of $\mathcal{O}_2$, respectively, and $\Pi_{\tau_1}=\Pi_{\sigma_1}|_{\mathcal{O}_\infty}$ with respect to a certain embedding of $\mathcal{O}_\infty$ into $\mathcal{O}_2$. © *2009 American Institute of Physics.* [DOI: 10.1063/1.3081386]

I. INTRODUCTION

The purpose of this paper is to show a new relation between dynamical systems and operator algebras. The former means jump transformations in metric number theory and the latter does an embedding of Cuntz algebras. In this section, we show our motivation and main theorem.

A. Motivation

We explain our motivation in this subsection. Mathematical details will be shown in Secs. I B, I C, and III.

As is well known, metric number theory has originated as Gauss' problem about the asymptotic behavior of iterations of the regular continued fraction transformation. In modern times, the theory is formulated as a measure theoretical dynamical system.[6,13] For a measure space (X,μ), let F be a measurable map from X to X which may not be invertible. We call the triplet (X,μ,F) a *(measure theoretical) dynamical system*. For example, the *Gauss map (or the continued fraction transformation)* τ_1 on the closed interval $[0,1]$ defined by

$$\tau_1(x) \equiv \begin{cases} \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor & (x \neq 0), \\ 0 & (x = 0), \end{cases} \tag{1.1}$$

gives an important dynamical system $([0,1],\lambda,\tau_1)$ with respect to the Lebesgue measure λ where $\lfloor\cdot\rfloor$ denotes Gauss' symbol (or the floor function). The second and third authors have studied a generalization of the continued fraction transformation in metric number theory.[3]

On the other hand, the first author has studied representations of Cuntz algebras and Perron–Frobenius operators.[8] A dynamical system (X,μ,F) induces a representation $(L_2(X,\mu),\Pi_F)$ of a Cuntz algebra when (X,μ,F) satisfies a certain condition. Properties of $(L_2(X,\mu),\Pi_F)$ are characterized by (X,μ,F). As an application, such a construction is used to construct type III factor representations of Cuntz–Krieger algebras (Ref. 10, Sec. 1.3).

These two different studies have the following common problem in dynamical system.

Problem 1.1: *For a given map F on a set X, find an F-invariant (or F-preserving) measure μ of X, that is,*

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$$\mu(F^{-1}(E)) = \mu(E) \tag{1.2}$$

for every μ-measurable subset E of X.

For example, a solution of Gauss' problem is given by an invariant measure of the Gauss map,[6,13] and there exists a relation between an invariant measure and the representation of Cuntz–Krieger algebra arising from the dynamical system.[8]

The new idea in this paper is to show a relation between (the construction of) jump transformation and an embedding of $\mathcal{O}_\infty$ into $\mathcal{O}_2$, where $\mathcal{O}_\infty$ and $\mathcal{O}_2$ denote Cuntz algebras which will be explained in Sec. I B 1. Let F and G be two maps, and let Π_F and Π_G denote representations associated with them, respectively. We show that Π_G is the restriction of Π_F when G is a jump transformation of F (Theorem 1.5). Especially, the Gauss map τ_1 and the Farey map σ_1 induce representations Π_{τ_1} of $\mathcal{O}_\infty$ and that Π_{σ_1} of $\mathcal{O}_2$, respectively, and $\Pi_{\tau_1}=\Pi_{\sigma_1}|_{\mathcal{O}_\infty}$ with respect to a certain embedding of $\mathcal{O}_\infty$ into $\mathcal{O}_2$ (Sec. III B).

B. Representations of Cuntz algebras arising from dynamical systems

In this subsection, we explain representations of Cuntz algebras arising from dynamical systems according to Ref. 8.

1. Cuntz algebras

For $N=2,3,\ldots,+\infty$, let $\mathcal{O}_N$ denote the *Cuntz algebra*,[4] that is, a C*-algebra which is universally generated by $s_1,\ldots,s_N$ satisfying

$$s_i^* s_j = \delta_{ij} I \quad \text{for} \quad i,j = 1,\ldots,N, \tag{1.3}$$

$$\sum_{i=1}^{N} s_i s_i^* = I \quad (\text{if } N < +\infty), \tag{1.4}$$

$$\sum_{i=1}^{k} s_i s_i^* \leq I \quad \text{for} \quad k = 1,2,\ldots (\text{if } N = +\infty), \tag{1.5}$$

where I denotes the unit of $\mathcal{O}_N$.

Since $\mathcal{O}_N$ is simple, that is, there is no nontrivial closed two-sided ideal, any unital homomorphism from $\mathcal{O}_N$ to a C*-algebra is injective. If $t_1,\ldots,t_N$ are elements of a unital C*-algebra $\mathfrak{A}$ such that $t_1,\ldots,t_N$ satisfy the relations of canonical generators of $\mathcal{O}_N$, then the correspondence $s_i \mapsto t_i$ for $i=1,\ldots,N$ is uniquely extended to a $*$-embedding of $\mathcal{O}_N$ into $\mathfrak{A}$ from the uniqueness of $\mathcal{O}_N$. Therefore we call such a correspondence among generators by an embedding of $\mathcal{O}_N$ into $\mathfrak{A}$.

Assume that $s_1,\ldots,s_N$ are realized as operators on a Hilbert space $\mathcal{H}$. According to (1.3) and (1.4), $\mathcal{H}$ is decomposed into orthogonal subspaces as $s_1\mathcal{H}\oplus\cdots\oplus s_N\mathcal{H}$. Since s_i is an isometry, $s_i\mathcal{H}$ has the same dimension as $\mathcal{H}$. From this, we see that there is no finite dimensional representation of $\mathcal{O}_N$ which preserves the unit. The illustration (Fig. 1) is helpful in understanding $s_1,\ldots,s_N$.

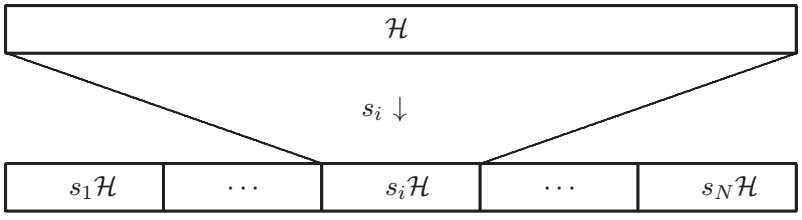

FIG. 1. Action of the Cuntz algebra O_N on a Hilbert space N.

2. Representations of Cuntz algebras

We introduce (Ref. 1, Chap. 2). Let (X,μ) be a measure space and $\Omega_N \equiv \{1,\ldots,N\}$ for $2 \leq N < \infty$ and $\Omega_\infty \equiv \mathbf{N} = \{1,2,3,\ldots\}$. Let F be a measurable map on (X,μ). Define the new measure $\mu \circ F$ on X by $(\mu \circ F)(E) \equiv \mu(F(E))$ for $E \subset X$. For $2 \leq N \leq \infty$, a family $f = \{f_i : i \in \Omega_N\}$ of maps on X is a *branching function system if*

(i) f_i is a measurable map from X to X for each i,
(ii) if $A_i \equiv f_i(X)$, then $\mu(X \setminus \cup_{i \in \Omega_N} A_i) = 0$ and $\mu(A_i \cap A_j) = 0$ when $i \neq j$,
(iii) there exists the Radon–Nikodym derivative Φ_{f_i} of $\mu \circ f_i$ with respect to μ and $\Phi_{f_i} > 0$ almost everywhere X for each i.

A map F on X is called the *coding map* of a branching function $f = \{f_i : i \in \Omega_N\}$ if $F \circ f_i = id_X$ for each i. By definition, F is measurable and Φ_F exists.

Let $L_2(X,\mu)$ denote the Hilbert space of all square integrable complex valued functions on X. For a branching function system $f = \{f_i : i \in \Omega_N\}$ with the coding map F, define the family $\{S(f_i) : i \in \Omega_N\}$ of operators on $L_2(X,\mu)$ by

$$\{S(f_i)\phi\}(x) \equiv \chi_{A_i}(x) \cdot \{\Phi_F(x)\}^{1/2} \cdot \phi(F(x)) \quad (\phi \in L_2(X,\mu), x \in X), \tag{1.6}$$

where χ_{A_i} denotes the characteristic function of A_i. From this, adjoint operators $\{S(f_i)^* : i \in \Omega_N\}$ are given as follows:

$$\{S(f_i)^*\phi\}(x) = \{\Phi_{f_i}(x)\}^{1/2} \cdot \phi(f_i(x)) \quad (\phi \in L_2(X,\mu), x \in X). \tag{1.7}$$

Then we can verify that $S(f_i)$ is an isometry for each i.

Lemma 1.2: *For* $i,j \in \Omega_N$, *define* $(f_i \circ f_j)(x) \equiv f_i(f_j(x))$. *Then the following holds:*

$$S(f_i)S(f_j) = S(f_i \circ f_j) \quad (i,j \in \Omega_N). \tag{1.8}$$

Proof: Let $\phi \in L_2(X,\mu)$. By (1.6),

$$\{S(f_i)S(f_j)\phi\}(x) = \chi_{A_i}(x)\{\Phi_F(x)\}^{1/2}\chi_{A_j}(F(x))\{\Phi_F(F(x))\}^{1/2}\phi(F^2(x)) \tag{1.9}$$

for $x \in X$ and $i,j \in \Omega_N$. On the other hand, we see that $\{f_i \circ f_j : i,j \in \Omega_N\}$ is also a branching function system on X with the coding map F^2 by identifying $\Omega_N \times \Omega_N$ with Ω_{N^2}, where $F^2 \equiv F \circ F$. Hence the operator $S(f_i \circ f_j)$ is well defined which is given as follows:

$$\{S(f_i \circ f_j)\phi\}(x) = \chi_{(f_i \circ f_j)(X)}(x) \cdot \{\Phi_{F^2}(x)\}^{1/2}\phi(F^2(x)) \tag{1.10}$$

for $x \in X$ and $i,j \in \Omega_N$. In (1.9), we can directly verify that $\chi_{A_i}(x)\chi_{A_j}(F(x)) = \chi_{(f_i \circ f_j)(X)}(x)$ and $\Phi_F(x)\Phi_F(F(x)) = \Phi_{F^2}(x)$ for almost all $x \in X$. From these and (1.10), the statement holds. ■

Let $\{s_i : i \in \Omega_N\}$ denote canonical generators of $\mathcal{O}_N$. For a branching function system $f = \{f_i : i \in \Omega_N\}$,

$$\pi_f(s_i) \equiv S(f_i) \quad (i \in \Omega_N) \tag{1.11}$$

defines a representation $(L_2(X,\mu), \pi_f)$ of the Cuntz algebra $\mathcal{O}_N$ because $\{S(f_i) : i \in \Omega_N\}$ satisfy relations of canonical generators of $\mathcal{O}_N$. Especially, branching function system $f = \{f_1, f_2\}$ and $g = \{g_i : i \in \mathbf{N}\}$ define a representation of $\mathcal{O}_2$ and that of $\mathcal{O}_\infty$, respectively. Examples will be shown in Sec. III.

Remark 1.3: For a dynamical system (X,μ,F), a branching function system with the coding map F is not unique even if it exists. Therefore the representation π of $\mathcal{O}_N$ arising from (X,μ,F) depends on the choice of a branching function system with the coding map F.

C. Main theorem

In this subsection, we show our main theorem. For this purpose, we introduce jump transformation according to Ref. 13.

Definition 1.4: *Let (X,μ,F) be a dynamical system. Assume that A is a measurable subset of X such that*

$$\mu(X \setminus \bigcup_{n\geq 1} F^{-n}(A)) = 0. \tag{1.12}$$

(i) *For the pair (F,A), the map e from X to $\mathbf{Z}_{\geq 0} \equiv \{0,1,2,3,\ldots\}$ is called the first entry time of (F,A) if*

$$e(x) \equiv \min\{k \in \mathbf{Z}_{\geq 0} : T^k(x) \in A\} \quad (x \in X). \tag{1.13}$$

(ii) *For the pair (F,A), define the map $J_{F,A}$ from X to X by*

$$J_{F,A}(x) \equiv F^{e(x)+1}(x) \quad (x \in X). \tag{1.14}$$

(iii) *We call $J_{F,A}$ the jump transformation of (F,A).*

We briefly explain the meaning of jump transformation as follows. The assumption (1.12) means that it surely takes finite time such that x enters the subset A with respect to the discrete time evolution

$$x \mapsto F(x) \tag{1.15}$$

for almost all $x \in X$. From this explanation, we can also understand the meaning of the first entry time. For $x \in X$, assume $n \equiv e(x) < \infty$. Then the transformation $J_{F,A}$ maps x in Fig. 2.

Let $\{t_1,t_2\}$ and $\{s_n : n \in \mathbf{N}\}$ denote canonical generators of Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, respectively. Assume that $\mathcal{O}_\infty$ is embedded into $\mathcal{O}_2$ by

$$s_n = t_2^{n-1} t_1 \quad (n \geq 1), \tag{1.16}$$

where we write $t_2^0 \equiv I$ for convenience. This embedding induces the restriction of representations of $\mathcal{O}_2$ on $\mathcal{O}_\infty$,

$$\mathrm{Rep}\,\mathcal{O}_2 \ni \pi \mapsto \pi|_{\mathcal{O}_\infty} \in \mathrm{Rep}\,\mathcal{O}_\infty.$$

Theorem 1.5: *Let (X,μ,F) be dynamical system such that F is the coding map of a branching function system $f=\{f_1,f_2\}$. Assume that $A_1 \equiv f_1(X)$ satisfies (1.12) with respect to (X,μ,F). Let π_f denote the representation of $\mathcal{O}_2$ associated with f in (1.11). Let J_{F,A_1} be as in (1.14) for (F,A_1). Then the following holds.*

(i) *There exists a branching function system $g=\{g_n : n \in \mathbf{N}\}$ on (X,μ) with the coding map J_{F,A_1} such that the representation π_g of $\mathcal{O}_\infty$ coincides with the restriction $\pi_f|_{\mathcal{O}_\infty}$ of π_f on $\mathcal{O}_\infty$ with respect to the embedding in (1.16), that is,*

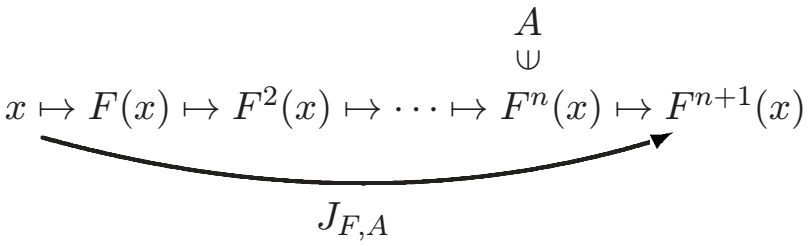

FIG. 2. Jump transformation.

$$\pi_f|_{\mathcal{O}_\infty} = \pi_g. \tag{1.17}$$

(ii) *If ϕ is the density of an invariant measure with respect to F, that is,*

$$\nu(E) \equiv \int_E \phi(x) d\mu(x) \quad (E \subset X) \tag{1.18}$$

defines an F-invariant measure ν of X and ϕ belongs to $L_1(X,\mu)$, then the function ψ on X defined by

$$\psi(x) = \{\{\pi_f(t_1)^* \sqrt{\phi}\}(x)\}^2 \quad (x \in X) \tag{1.19}$$

is the density of an invariant measure of J_{F,A_1}, where $\sqrt{\phi}(x) \equiv \sqrt{\phi(x)}$.

We summarize Theorem 1.5, part (i) as follows:

Map	Cuntz algebra
F	Representation of $\mathcal{O}_2 = C^*\langle\{t_1,t_2\}\rangle$
G	Representation of $\mathcal{O}_\infty = C^*\langle\{s_n : n \in \mathbf{N}\}\rangle$
$G = J_{F,A_1}$	$s_n = t_2^{n-1} t_1$

Remark 1.6:

(i) We stress that the jump transformation (1.14) and the embedding (1.16) are independently introduced in different studies. Theorem 1.5 is a casual discovery. We see that the index $e(x)+1$ of $F^{e(x)+1}(x)$ and the index $n-1$ of the element t_2 of the equation $s_n = t_2^{n-1} t_1$ fit like a glove in the proof of Theorem 1.5. This is an interesting accidental coincidence.

(ii) From Theorem 1.5, we see that a jump transformation gives a geometric realization of the embedding of $\mathcal{O}_\infty$ into $\mathcal{O}_2$ in (1.16). On the other hand, the embedding in (1.16) appears in the construction of a unitary isomorphism between the Bose–Fock space and the Fermi–Fock space.[9] From this, we suspect the existence of a behind mathematical structure among dynamical system, operator algebra, and quantum field theory.

(iii) Apparently, an embedding of $\mathcal{O}_\infty$ into $\mathcal{O}_2$ is not unique. For example, the following is one of other embeddings of $\mathcal{O}_\infty$ into $\mathcal{O}_2$:

$$s_n = t_2^{n-1}(t_1 t_2 t_1^* + t_1^2 t_2^*) \quad (n \geq 1). \tag{1.20}$$

Therefore the choice of the embedding (1.16) is also an interesting accident.

(iv) It is well known that there are many other natural embeddings between Cuntz algebras, for example, $\mathcal{O}_3 \hookrightarrow \mathcal{O}_2$. Hence one can expect that similar results hold for all such cases, mutatis mutandis. However, we cannot find a jump transformation for such embeddings yet. About relations between embeddings and representations, refer to Ref. 7.

In Sec. II, we prove Theorem 1.5. In Sec. III, we show examples of Theorem 1.5.

II. PROOF OF THEOREM

We prove Theorem 1.5 in this section.

Theorem 2.1: (Reference 13, Sec. 19.2.1) Let (X,μ,F) be a dynamical system and assume that μ is F-invariant and a measurable subset A of X satisfies (1.12). Define the new measure ν by

$$\nu(E) \equiv \mu(F^{-1}(E) \cap A) \quad (E \subset X). \tag{2.1}$$

Then ν is invariant with respect to the jump transformation of (F,A).

Lemma 2.2: *Let (X,μ,F) be a dynamical system, and let ν_1 and ν_2 be other measures on X. Assume that ν_1, ν_2 and $\mu \circ F$ are absolutely continuous with respect to μ. We write $\psi_i \equiv d\nu_i/d\mu$ for $i=1,2$ and $\Phi_F \equiv d(\mu \circ F)/d\mu$. Then the following are equivalent:*

(i) $\nu_1 = \nu_2 \circ F$.
(ii) $\psi_1 = \Phi_F \cdot (\psi_2 \circ F)$ *almost everywhere in* X.

Proof: By using the chain rule of Radon–Nikodym derivatives, the statement holds.

From Lemma 2.2, the following holds.

Corollary 2.3: *For* $a,b \in \mathbf{R}$, $a<b$, *let* $[a,b]$ *and* (a,b) *denote the closed interval and the open interval. Let* F *be a piecewise differentiable map from* (a,b) *to* (a,b) *with the differential* F' *and let* ν_1 *and* ν_2 *be measures on* $[a,b]$. *Assume that* ν_1, ν_2 *and* $\lambda \circ F$ *are absolutely continuous with respect to the Lebesgue measure* λ *on* $[a,b]$. *Define* $\psi_i \equiv d\nu_i/d\lambda$ *for* $i=1,2$. *Then the following are equivalent:*

(i) $\nu_1 = \nu_2 \circ F$.
(ii) $\psi_1 = |F'| \cdot (\psi_2 \circ F)$ *almost everywhere in* $[a,b]$, *where* $|F'|(x) \equiv |F'(x)|$ *for* $x \in (a,b)$.

Proof of Theorem 1.5:

(i) Define the branching function system $g=\{g_n : n \in \mathbf{N}\}$ on X by

$$g_n \equiv f_2^{n-1} \circ f_1 \quad (n \geq 1). \tag{2.2}$$

Since f is a branching function system, we can verify that g is a branching function system on X. From (1.16) and (1.8),

$$\pi_f(s_n) = \pi_f(t_2)^{n-1}\pi_f(t_1) = S(f_2)^{n-1}S(f_1) = S(f_2^{n-1} \circ f_1) = S(g_n) = \pi_g(s_n).$$

Hence (1.17) holds. If G denotes the coding map of g, then

$$G(x) = g_n^{-1}(x) = (f_1^{-1} \circ f_2^{-n+1})(x) \quad \text{when } x \in g_n(X), \quad n \in \mathbf{N}. \tag{2.3}$$

Remark $g_n(X) = f_2^{n-1}(A_1)$ for each $n \in \mathbf{N}$. On the other hand, when $x \in g_n(X)$, $e(x) = n-1$. From this, J_{F,A_1} satisfies that

$$J_{F,A_1}(x) = F^n(x) = (f_1^{-1} \circ f_2^{-n+1})(x). \tag{2.4}$$

From (1.14), $G = J_{F,A_1}$ holds almost everywhere in X. Hence the statement holds.

(ii) Define measures ν_∞ and ν_2 of X by

$$\nu_\infty(E) \equiv \int_E \psi(x)d\mu(x), \quad \nu_2(E) \equiv \int_E \phi(x)d\mu(x) \tag{2.5}$$

for each measurable subset E of (X,μ). From (1.19) and the definition of π_f,

$$\psi(x) = \{\{S(f_1)^*\sqrt{\phi}\}(x)\}^2 = \Phi_{f_1}(x) \cdot \phi(f_1(x)). \tag{2.6}$$

From Lemma 2.2,

$$\nu_\infty(E) = (\nu_2 \circ f_1)(E) = \nu_2(F^{-1}(E) \cap A_1). \tag{2.7}$$

From Theorem 2.1, the statement holds. ■

III. EXAMPLES

We show examples of Theorem 1.5 in this section. For convenience, we show formulas of representations as follows. Assume that $X=[a,b]$ and F is a piecewise differentiable map on X which is the coding map of a branching function system $\{f_i : i \in \Omega_N\}$ on X for $2 \leq N \leq \infty$. Then (1.6) is given as follows:

$$\{\pi_f(s_i)\phi\}(x) = \chi_{A_i}(x)\sqrt{|F'(x)|}\,\phi(F(x)), \tag{3.1}$$

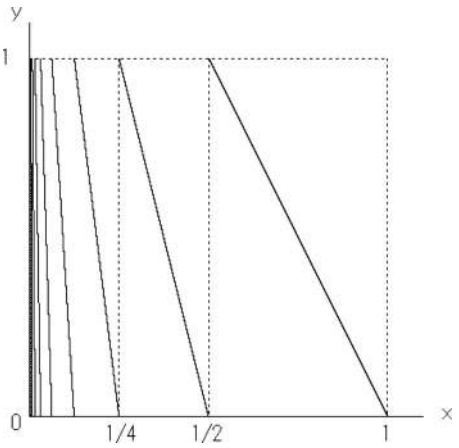

FIG. 3. Jump transformation of tent map.

$$\{\pi_f(s_i^*)\phi\}(x) = \sqrt{|f_i'(x)|}\,\phi(f_i(x)) \tag{3.2}$$

for $i \in \Omega_N$.

A. Tent map

As an introductory example of jump transformation, we show a tent map. Define the map Λ by $\Lambda(x) \equiv 1 - |2x-1|$ on $x \in [0,1]$. The map Λ is called a *tent map*.[5] Let $A_1 \equiv [1/2,1]$, and let $f_2 \equiv (\Lambda|_{[0,1/2]})^{-1}$ and $f_1 \equiv (\Lambda|_{[1/2,1]})^{-1}$. Then

$$f_1(x) = 1 - 1/2x, \quad f_2(x) = 1/2x. \tag{3.3}$$

Define $g_n \equiv f_2^{n-1} \circ f_1$ for $n \geq 1$. Then

$$g_n(x) = \frac{2-x}{2^n}. \tag{3.4}$$

The coding map G of $\{g_n : n \in \mathbf{N}\}$ is given by

$$G(x) = 2 - 2^n x \quad \text{when} \quad 1/2^n < x \leq 1/2^{n-1}. \tag{3.5}$$

Hence the jump transformation is given as follows (Figs. 3 and 4):

$$J_{\Lambda,A_1}(x) = G(x) = 2 - 2^{\lfloor \log_2 x^{-1} \rfloor + 1} \cdot x \quad (x \in (0,1]). \tag{3.6}$$

Both of these invariant probability measures are the Lebesgue measure. For $f=\{f_1,f_2\}$ in (3.3) and $g=\{g_n : n \in \mathbf{N}\}$ in (3.4), the representations $(L_2[0,1],\pi_f)$ of $\mathcal{O}_2$ and that $(L_2[0,1],\pi_g)$ of $\mathcal{O}_\infty$ are given as follows:

$$\{\pi_f(t_1)\phi\}(x) = \chi_{[1/2,1]}(x)\sqrt{2}\,\phi(\Lambda(x)),$$

$$\{\pi_f(t_2)\phi\}(x) = \chi_{[0,1/2]}(x)\sqrt{2}\,\phi(\Lambda(x)), \tag{3.7}$$

$$\{\pi_g(s_n)\phi\}(x) = \chi_{[2^{-n},2^{-(n-1)}]}(x)2^{n/2}\phi(G(x)) \quad (n \geq 1). \tag{3.8}$$

From (1.17), $\pi_f|_{\mathcal{O}_\infty} = \pi_g$.

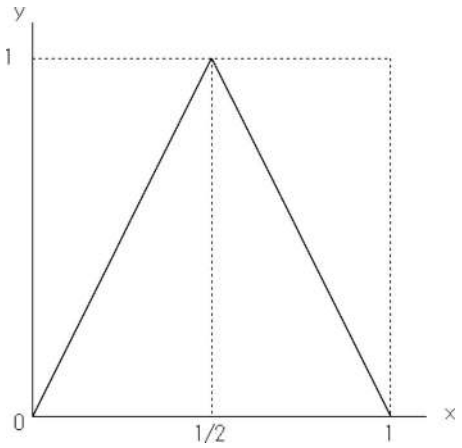

FIG. 4. Tent map.

B. Gauss map and Farey map

It is well known that the Gauss map τ_1 in (1.1) is the jump transformation of the *Farey map* σ_1,[11,13,14] which is defined on [0,1] by (Fig. 5 and 6)

$$\sigma_1(x) \equiv \frac{2}{1+|1-2x|} - 1, \tag{3.9}$$

where $A_1 \equiv [1/2, 1]$.

Define two measures γ and θ on [0,1] by

$$d\gamma(x) \equiv \frac{1}{\log 2} \frac{dx}{x+1}, \tag{3.10}$$

$$d\theta(x) \equiv \frac{dx}{x}. \tag{3.11}$$

The measure γ is called *Gauss' measure* (Ref. 6, p. 16). Then γ and θ are invariant measures of τ_1 and σ_1, respectively,[14] where dx denotes the Lebesgue measure on [0,1]. Remark that the former is finite but the latter is not.

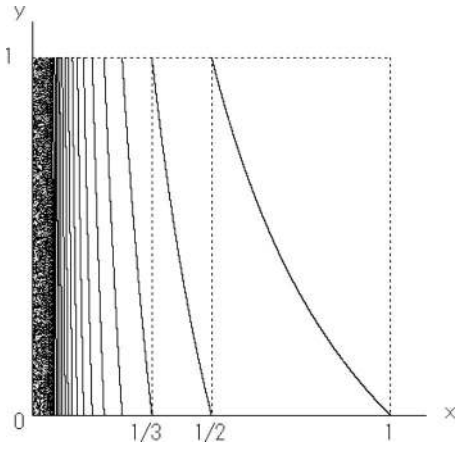

FIG. 5. Gauss map.

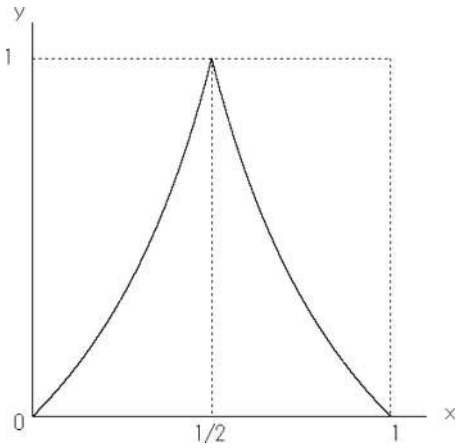

FIG. 6. Farey map.

For $k \geq 1$, define $g_k \equiv (\tau_1)^{-1}$ on $1/(k+1) < x \leq 1/k$, and $f_1 \equiv (\sigma_1|_{[1/2,1]})^{-1}$ and $f_2 \equiv (\sigma_1|_{[0,1/2]})^{-1}$. Then $\{g_k : k \in \mathbf{N}\}$ and $\{f_1, f_2\}$ are branching function systems on $[0,1]$ with the coding maps τ_1 and σ_1, respectively. Furthermore,

$$g_k(x) = \frac{1}{x+k} \quad (k \geq 1), \tag{3.12}$$

$$f_1(x) = \frac{1}{x+1}, \quad f_2(x) = \frac{x}{x+1} \tag{3.13}$$

for $x \in [0,1]$. We see that $g_k = f_2^{k-1} \circ f_1$ for $k \geq 1$. From these, we obtain representations π_{τ_1} and π_{σ_1} of $\mathcal{O}_\infty$ and $\mathcal{O}_2$ on $L_2[0,1]$, respectively, which are written as follows:

$$\{\pi_{\tau_1}(s_n)\phi\}(x) = \frac{\chi_{[1/(n+1),1/n]}(x)}{x}\phi(\tau_1(x)) \quad (n \in \mathbf{N}), \tag{3.14}$$

$$\{\pi_{\sigma_1}(t_1)\phi\}(x) = \frac{\chi_{[1/2,1]}(x)}{x}\phi(\sigma_1(x)),$$

$$\{\pi_{\sigma_1}(t_2)\phi\}(x) = \frac{\chi_{[0,1/2]}(x)}{1-x}\phi(\sigma_1(x)) \tag{3.15}$$

for $x \in [0,1]$ and $\phi \in L_2[0,1]$.

Remark 3.1: Remark that the invariant measure θ of the Farey map in (3.11) is not finite. Therefore Theorem 1.5, part (ii) does not seem to apply to this case. However, if one forgets that the operator $\pi_{\sigma_1}(t_1)^*$ is defined as an operator from $L_2[0,1]$ to $L_2[0,1]$, then we see that Theorem 1.5, part (ii) applies also to this case as follows. By definition,

$$\{\pi_{\sigma_1}(t_1)^*\phi\}(x) = \frac{1}{x+1}\phi\left(\frac{1}{x+1}\right) \quad (\phi \in L_2[0,1], x \in [0,1]). \tag{3.16}$$

Although the density $\rho_0(x) \equiv 1/x$ of θ does not belong to $L_1[0,1]$,

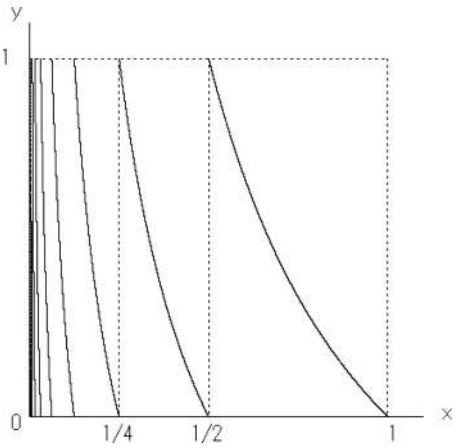

FIG. 7. τ_2.

$$\{\{\pi_{\sigma_1}(t_1)^*\sqrt{\rho_0}\}(x)\}^2 = \left\{\frac{1}{x+1}\sqrt{\frac{1}{\frac{1}{x+1}}}\right\}^2 = \frac{1}{x+1} \quad (x \in [0,1]). \tag{3.17}$$

Therefore the $\{\{\pi_{\sigma_1}(t_1)^*\sqrt{\rho_0}\}(x)\}^2$ is the density of Gauss' measure γ in (3.10) up to scalar multiple. Hence Theorem 1.5, part (ii) holds for this case.

We summarize relations between maps and Cuntz algebras in this example as follows:

Map	Cuntz algebra
Gauss map	Representation of $\mathcal{O}_\infty$
Farey map	Representation of $\mathcal{O}_2$
Jump transformation	Embedding of $\mathcal{O}_\infty$ into $\mathcal{O}_2$

C. A generalization of continued fraction transformation

In this subsection, we show an example of jump transformation associated with a generalization of continued fraction transformation. Define the map τ_2 on $[0,1]$ by (Fig. 7)

$$\tau_2(x) = \begin{cases} \dfrac{1}{2^{k-1}x} - 1 & (2^{-k} < x \le 2^{-k+1}, k \in \mathbf{N}) \\ 0 & (x = 0). \end{cases} \tag{3.18}$$

The map τ_2 was introduced by Chan.[2] The invariant probability measure μ_2 of τ_2 is given as follows [Ref. 2, (2.20)):

$$d\mu_2(x) = \frac{1}{\log\frac{4}{3}}\frac{dx}{(x+1)(x+2)} \quad (x \in [0,1]). \tag{3.19}$$

For $k \in \mathbf{N}$, define $g_k \equiv (\tau_2|_{X_k})^{-1}$ on $X_k \equiv (2^{-k}, 2^{-k+1}]$. Then

$$g_k(x) = \frac{1}{2^{k-1}(x+1)} \quad (x \in [0,1)). \tag{3.20}$$

Define the map σ_2 from $[0,1]$ to $[0,1]$ by (Fig. 8)

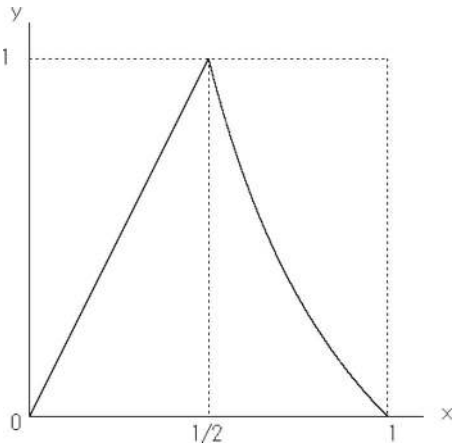

FIG. 8. σ_2.

$$\sigma_2(x) \equiv \begin{cases} 2x & \left(0 \le x \le \frac{1}{2}\right), \\ \dfrac{1}{x} - 1 & \left(\frac{1}{2} \le x \le 1\right). \end{cases} \tag{3.21}$$

Then τ_2 is the jump transformation of (σ_2, A_1) for $A_1 \equiv [1/2, 1]$, and Gauss' measure γ in (3.10) is σ_2-invariant.

Define maps f_1 and f_2 on $[0,1]$ by

$$f_1(x) \equiv \frac{1}{x+1}, \quad f_2(x) \equiv \frac{1}{2}x. \tag{3.22}$$

Then $\{f_1, f_2\}$ is a branching function system on $[0,1]$ with the coding map σ_2. We see that

$$g_k(x) = f_2^{k-1} \circ f_1 \quad (k \ge 1). \tag{3.23}$$

For $f=\{f_1, f_2\}$ in (3.22), the representation π_f of $\mathcal{O}_2$ on $L_2[0,1]$ is given by

$$\{\pi_f(t_1)\phi\}(x) \equiv \chi_{[1/2,1]}(x)(1+x)\phi\left(\frac{1}{x} - 1\right),$$

$$\{\pi_f(t_2)\phi\}(x) \equiv \chi_{[0,1/2]}(x)\sqrt{2}\phi(2x) \tag{3.24}$$

for $\phi \in L_2[0,1]$ and $x \in [0,1]$. From this,

$$\{\pi_f(t_1^*)\phi\}(x) = \frac{1}{1+x}\phi\left(\frac{1}{1+x}\right),$$

$$\{\pi_f(t_2^*)\phi\}(x) = \frac{1}{\sqrt{2}}\phi\left(\frac{1}{2}x\right). \tag{3.25}$$

Define $\phi_0(x) \equiv (1/\log 2)(1/x+1)$. From Theorem 5, part (ii),

$$\psi(x) = \{\{\pi(t_1^*)\sqrt{\phi_0}\}(x)\}^2 = \frac{1}{(1+x)^2}\frac{1}{1+\frac{1}{1+x}} = \frac{1}{(1+x)(2+x)}. \tag{3.26}$$

In this way, the density ψ of τ_2-invariant measure μ_2 in (3.19) is given up to scalar multiple.

For further generalizations of τ_1 and τ_2, see Ref. 3. About operator theory associated with continued fractions, see Refs. 6, 12, and 13.

ACKNOWLEDGMENTS

The authors would like to express their sincere thanks to Yasushi Kogure, and especially to the referees for their numerous suggestions.

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Journal of Number Theory 129 (2009) 3069–3080

Contents lists available at ScienceDirect

Journal of Number Theory

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Continued fraction expansions and permutative representations of the Cuntz algebra $\mathcal{O}_\infty$

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ARTICLE INFO

Article history:
Received 15 January 2009
Revised 18 June 2009
Available online 13 August 2009
Communicated by David Goss

MSC:
11A55
46K10

Keywords:
Cuntz algebra
Permutative representation
Continued fraction expansion

ABSTRACT

We show a correspondence between simple continued fraction expansions of irrational numbers and irreducible permutative representations of the Cuntz algebra $\mathcal{O}_\infty$. In regard to the correspondence, it is shown that the equivalence of real numbers with respect to modular transformations is equivalent to the unitary equivalence of representations. Furthermore, we show that quadratic irrationals are related to irreducible permutative representations of $\mathcal{O}_\infty$ with a cycle.

1. Introduction

The purpose of this paper is to show a new relation between number theory and the representation theory of operator algebras. At the beginning, we show our motivation. Explicit mathematical statements will be given after Section 1.2. The main theorems will be shown in Section 1.4.

1.1. Motivation

Continued fractions furnish important tools in number theory [9,11,16], and continued fraction transformations induce typical dynamical systems [3,4,10,17]. It is well-known that the dynamical system of the simple continued fraction transformation on the set of irrational numbers in the interval

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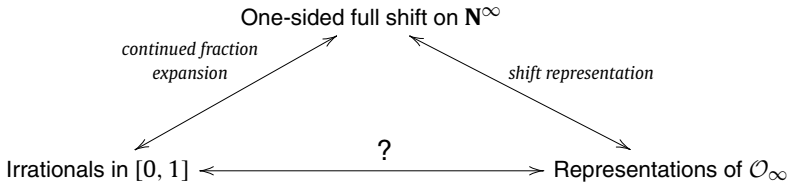
0022-314X/$ – see front matter
doi:10.1016/j.jnt.2009.06.003

$[0,1]$ is conjugate with the one-sided full shift on the set $\mathbf{N}^{\infty} \equiv \{(n_i)_{i\geqslant 1}\colon n_i \in \mathbf{N} \text{ for all } i\}$ [15] by using continued fraction expansions where $\mathbf{N} \equiv \{1,2,3,\ldots\}$.

On the other hand, such dynamical systems induce representations of Cuntz algebras and their relations were studied [12,14]. For example, the one-sided full shift on $\mathbf{N}^{\infty}$ induces the shift representation of $\mathcal{O}_{\infty}$, which acts on the representation space $l_2(\mathbf{N}^{\infty})$ [2]. The shift representation is decomposed into the direct sum of irreducibles unique up to unitary equivalence and the decomposition is multiplicity free. Every irreducible component in the decomposition is a permutative representation, and any irreducible permutative representation appears in the decomposition.

From these facts, we are interested in a relation between continued fraction expansions of irrationals and such representations of $\mathcal{O}_{\infty}$ by the intermediary of the one-sided full shift on $\mathbf{N}^{\infty}$. We roughly illustrate relations among theories as follows:



The position in the above question mark is the content of this study.

1.2. Continued fraction expansion map and continued fraction transformation on the set of irrationals

We review the continued fraction expansion map and the continued fraction transformation according to [9]. Let $[0,1]$ denote the closed interval from 0 to 1, and let Ω denote the set of all irrationals in $[0,1]$, that is,

$$\Omega = [0,1] \setminus \mathbf{Q}. \tag{1.1}$$

Remark that we consider only Ω but not the whole of $[0,1]$ in this paper. Any $x \in \Omega$ has a unique infinite continued fraction expansion [9, Theorem 170], that is, there exists a unique infinite sequence $(a_i(x))_{i\geqslant 1}$ of positive integers such that

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cfrac{1}{\ddots}}}}. \tag{1.2}$$

From this, we define the map CFE from Ω to $\mathbf{N}^{\infty}$ by

$$\mathsf{CFE}(x) \equiv \big(a_i(x)\big)_{i\geqslant 1}. \tag{1.3}$$

It is known that the map CFE is bijective [9, Theorems 161, 166, 169]. We call CFE *the continued fraction expansion map*. If $x \in \Omega$ is a solution of a quadratic equation with integral coefficients, then we call x a *quadratic irrational*.

Fact 1.1. (See [9, Theorem 177].) If $x \in \Omega$ is a quadratic irrational, then there exist a finite sequence $(m_1,\ldots,m_l) \in \mathbf{N}^l \cup \{\emptyset\}$ and a nonperiodic finite sequence $(n_1,\ldots,n_k) \in \mathbf{N}^k$ such that

$$\mathsf{CFE}(x) = (m_1,\ldots,m_l,n_1,\ldots,n_k,n_1,\ldots,n_k,n_1,\ldots,n_k,\ldots). \tag{1.4}$$

In Fact 1.1, we call $(n_1, \ldots, n_k)$ the *repeating block* of $\mathrm{CFE}(x)$ [16, Chapter 3] where we suppose that there is no shorter such repeating block and that the initial block does not end with a copy of the repeating block. Hence the repeating block of $\mathrm{CFE}(x)$ is uniquely defined for each quadratic irrational x in Ω. The converse of Fact 1.1 is also true [9, Theorem 176].

1.3. Permutative representations of $\mathcal{O}_\infty$

We review permutative representations of the Cuntz algebra $\mathcal{O}_\infty$ in this subsection.

1.3.1. $\mathcal{O}_\infty$

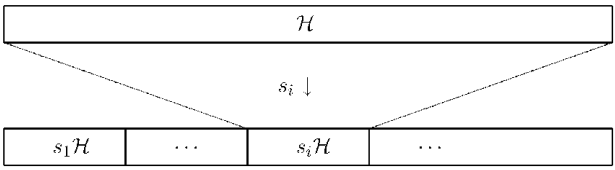
Let $\mathcal{O}_\infty$ denote the *Cuntz algebra* [6], that is, a C*-algebra which is universally generated by $\{s_i\colon\ i \in \mathbf{N}\}$ satisfying

$$s_i^* s_j = \delta_{ij} I \quad (i, j \in \mathbf{N}), \qquad \sum_{i=1}^{k} s_i s_i^* \leqslant I \quad (k \in \mathbf{N}), \tag{1.5}$$

where I denotes the unit of $\mathcal{O}_\infty$.

A *$*$-representation* of $\mathcal{O}_\infty$ is a pair $(\mathcal{H}, \pi)$ such that π is a $*$-homomorphism from $\mathcal{O}_\infty$ to the C*-algebra $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ [1]. We call $*$-representation as representation for simplicity of description. For two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ of $\mathcal{O}_\infty$, $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are *unitarily equivalent* if there exists a unitary u from $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that $u\pi_1(x)u^* = \pi_2(x)$ for each $x \in \mathcal{O}_\infty$. We state that a representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_\infty$ is *irreducible* if there is no invariant closed subspace of $\mathcal{H}$ except $\{0\}$ and $\mathcal{H}$; $(\mathcal{H}, \pi)$ is *multiplicity free* if any two subrepresentations of $(\mathcal{H}, \pi)$ are not unitarily equivalent. A representation $(\mathcal{H}, \pi)$ is irreducible if and only if the *commutant* $\{X \in \mathcal{L}(\mathcal{H})\colon\ X\pi(A) = \pi(A)X$ for all $A \in \mathcal{O}_\infty\}$ of $\pi(\mathcal{O}_\infty)$ equals to $\mathbf{C}I$.

Since $\mathcal{O}_\infty$ is simple, that is, there is no nontrivial closed two-sided ideal, any representation of $\mathcal{O}_\infty$ is injective. If $\{t_i\colon\ i \in \mathbf{N}\}$ are bounded operators on a Hilbert space $\mathcal{H}$ such that $\{t_i\colon\ i \in \mathbf{N}\}$ satisfy (1.5), then the correspondence $s_i \mapsto t_i$ for $i \in \mathbf{N}$ is uniquely extended to a unital $*$-representation of $\mathcal{O}_\infty$ on $\mathcal{H}$ from the uniqueness of $\mathcal{O}_\infty$. Therefore we call such a correspondence among generators by a representation of $\mathcal{O}_\infty$ on $\mathcal{H}$. Assume that $\{s_i\colon\ i \in \mathbf{N}\}$ are realized as operators on a Hilbert space $\mathcal{H}$. According to (1.5), $\mathcal{H}$ is decomposed into orthogonal subspaces as $\bigoplus_{i\in\mathbf{N}} s_i\mathcal{H}$. Since s_i is an isometry, $s_i\mathcal{H}$ has the same dimension as $\mathcal{H}$. From this, we see that there is no finite dimensional representation of $\mathcal{O}_\infty$ which preserves the unit. The following illustration is helpful in understanding $\{s_i\colon\ i \in \mathbf{N}\}$:



The algebra $\mathcal{O}_\infty$ appears in quantum field theory [13] and metrical number theory [14].

1.3.2. Permutative representations

We review permutative representations in this subsubsection.

Definition 1.2. (See [2,7,8,13].) Let $\{s_i\colon\ i \in \mathbf{N}\}$ denote the canonical generators of $\mathcal{O}_\infty$.

(i) A representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_\infty$ is permutative if there exists an orthonormal basis $\mathcal{E}$ $(\subset \mathcal{H})$ of $\mathcal{H}$ such that $\pi(s_i)\mathcal{E} \subset \mathcal{E}$ for each $i \in \mathbf{N}$.

(ii) For $J=(j_l)_{l=1}^{k}\in\mathbf{N}^k$ with $1\leqslant k<\infty$, let $P(J)$ denote the class of representations $(\mathcal{H},\pi)$ of $\mathcal{O}_\infty$ with a cyclic unit vector $\mathrm{v}\in\mathcal{H}$ such that $\pi(s_J)\mathrm{v}=\mathrm{v}$ and $\{\pi(s_{j_1}\cdots s_{j_k})\mathrm{v}\}_{l=1}^{k}$ is an orthonormal family in $\mathcal{H}$ where $s_J\equiv s_{j_1}\cdots s_{j_k}$.

(iii) For $J=(j_l)_{l\geqslant 1}\in\mathbf{N}^\infty$, let $P(J)$ denote the class of representations $(\mathcal{H},\pi)$ of $\mathcal{O}_\infty$ with a cyclic unit vector $\mathrm{v}\in\mathcal{H}$ such that $\{\pi(s_{J_{(n)}})^*\mathrm{v}\colon\ n\in\mathbf{N}\}$ is an orthonormal family in $\mathcal{H}$ where $J_{(n)}\equiv(j_1,\dots,j_n)$.

The vector v in both (ii) and (iii) is called the generalized permutative (= GP) vector of $(\mathcal{H},\pi)$.

We recall properties of these classes as follows: For any J, $P(J)$ in Definition 1.2(ii) and (iii) always exists and it is a class of permutative representations, which contains only one unitary equivalence class. From this, we can always identify $P(J)$ with a representative of $P(J)$ [2,7,8]. A representation $(\mathcal{H},\pi)$ of $\mathcal{O}_\infty$ is a *permutative representation with a cycle (chain)* if there exists $J\in\mathbf{N}^k$ for $1\leqslant k<\infty$ (*resp.* $J\in\mathbf{N}^\infty$) such that $(\mathcal{H},\pi)$ is $P(J)$. Details will be explained in Section 2.2.

1.4. Main theorems

We show our main theorems in this subsection. For this purpose, we construct two representations of $\mathcal{O}_\infty$ as follows. For a nonempty set A, let $l_2(A)$ denote the complex Hilbert space with an orthonormal basis $\{e_a\colon\ a\in A\}$. We call $\{e_a\colon\ a\in A\}$ the *standard basis* of $l_2(A)$.

Definition 1.3. Let Ω be as in (1.1) and let $\{s_i\colon\ i\in\mathbf{N}\}$ denote the canonical generators of $\mathcal{O}_\infty$.

(i) For $i\in\mathbf{N}$, define the map α_i from Ω to Ω by

$$\alpha_i(x)\equiv\frac{1}{x+i}\quad(x\in\Omega).\tag{1.6}$$

Define the representation π_α of $\mathcal{O}_\infty$ on $l_2(\Omega)$ by

$$\pi_\alpha(s_i)e_x\equiv e_{\alpha_i(x)}\quad(x\in\Omega,\ i\in\mathbf{N}).\tag{1.7}$$

(ii) For $i\in\mathbf{N}$, define the map β_i from $\mathbf{N}^\infty$ to $\mathbf{N}^\infty$ by

$$\beta_i(n_1,n_2,\dots)\equiv(i,n_1,n_2,\dots)\quad\big((n_1,n_2,\dots)\in\mathbf{N}^\infty,\ i\in\mathbf{N}\big).\tag{1.8}$$

Define the representation π_β of $\mathcal{O}_\infty$ on $l_2(\mathbf{N}^\infty)$ by

$$\pi_\beta(s_i)e_a\equiv e_{\beta_i(a)}\quad\big(a\in\mathbf{N}^\infty,\ i\in\mathbf{N}\big).\tag{1.9}$$

The representation $(l_2(\mathbf{N}^\infty),\pi_\beta)$ is called the *shift representation* [2].

Then the following holds.

Theorem 1.4. *Two representations $(l_2(\Omega),\pi_\alpha)$ and $(l_2(\mathbf{N}^\infty),\pi_\beta)$ are unitarily equivalent.*

From Theorem 1.4, we can compare irreducible components of $(l_2(\Omega),\pi_\alpha)$ with those of $(l_2(\mathbf{N}^\infty),\pi_\beta)$ and consider how an irreducible component in $(l_2(\Omega),\pi_\alpha)$ is realized. For this purpose, we recall a well-known equivalence relation of real numbers as follows.

Definition 1.5. (See [5,9,16].) If x and y are two real numbers such that

$$x = \frac{ay+b}{cy+d} \quad (a,b,c,d \in \mathbf{Z}) \tag{1.10}$$

where $ad-bc = \pm 1$, then x is said to be equivalent to y. In this case, we write $x \sim y$. The transformation (1.10) is called a modular transformation in a broad sense.

Remark that the transformation (1.10) for a,b,c,d with $ad-bc=1$ is also called a modular transformation in a narrow sense.

According to the unitary equivalence in Theorem 1.4, the following holds.

Theorem 1.6. *For $x \in \Omega$, let $[x]$ denote the equivalence class of x in Ω with respect to modular transformations, that is, $[x] = \{y \in \Omega : y \sim x\}$. Let $(l_2(\Omega), \pi_\alpha)$ be as in Definition* 1.3(i)*.*

(i) *The following irreducible decomposition holds*:

$$l_2(\Omega) = \bigoplus_{[x]\in\Omega/\sim} \mathcal{H}_{[x]} \tag{1.11}$$

where $\mathcal{H}_{[x]}$ denotes the closed subspace of $l_2(\Omega)$ generated by the subset $\{e_y : y \in [x]\}$ of $\{e_z : z \in \Omega\}$.

(ii) *For $x \in \Omega$, let $\eta_{[x]}$ denote the subrepresentation of π_α associated with the subspace $\mathcal{H}_{[x]}$ in* (1.11)*, that is,*

$$\eta_{[x]} \equiv \pi_\alpha|_{\mathcal{H}_{[x]}}. \tag{1.12}$$

Then $\eta_{[x]}$ and $\eta_{[y]}$ are unitarily equivalent if and only if $x \sim y$. Especially, (1.11) *is multiplicity free.*

(iii) *Let* CFE *be as in* (1.3) *and let $\Omega^{(2)}$ denote the set of all quadratic irrationals in Ω.*

(a) *If $x \in \Omega \setminus \Omega^{(2)}$, then $\eta_{[x]}$ is $P(\mathsf{CFE}(x))$.*

(b) *If $x \in \Omega^{(2)}$, then $\eta_{[x]}$ is $P(\mathsf{CFE}_0(x))$ where $\mathsf{CFE}_0(x)$ denotes the repeating block of* $\mathsf{CFE}(x)$.

(iv) *Any irreducible permutative representation of $\mathcal{O}_\infty$ is unitarily equivalent to $\eta_{[x]}$ for some $[x] \in \Omega/\sim$.*

In consequence, Theorem 1.6 shows that the set $\Omega/\sim$ of all equivalence classes of irrationals in $[0,1]$ is one-to-one correspondence in the set $\mathrm{IPR}(\mathcal{O}_\infty)/\sim$ of all unitary equivalence classes of irreducible permutative representations of $\mathcal{O}_\infty$:

$$\Omega/\sim \overset{\eta}{\cong} \mathrm{IPR}(\mathcal{O}_\infty)/\sim; \qquad [x] \mapsto \eta_{[x]} \tag{1.13}$$

where we identify $\eta_{[x]}$ with the unitary equivalence class of $\eta_{[x]}$ for convenience. Especially, the following equivalence holds as the restriction of η on the subset $\Omega^{(2)}/\sim$ of $\Omega/\sim$:

$$\Omega^{(2)}/\sim \overset{\eta}{\cong} \mathrm{IPR}_{cycle}(\mathcal{O}_\infty)/\sim \tag{1.14}$$

where $\mathrm{IPR}_{cycle}(\mathcal{O}_\infty)/\sim$ denotes the set of all unitary equivalence classes of irreducible permutative representations of $\mathcal{O}_\infty$ with a cycle.

Remark 1.7. The equivalence of numbers by modular transformations is well-known in number theory, such that the discriminant of an irrational is invariant with respect to modular transformations [5,18]. On the other hand, the unitary equivalence of representations is basic in the representation theory of $*$-algebras. Since these two equivalence relations are independently introduced in different mathematical areas, the equivalence of two equivalence relations in Theorem 1.6(ii) is nontrivial.

From Theorem 1.6, the following natural questions are thought up.

Problem 1.8.

(i) Show the meaning of the discriminant for the representation associated with a quadratic irrational. What are the discriminant and the class number [18, p. 59] in the representation theory of $\mathcal{O}_\infty$?
(ii) Find similar relations between the Cuntz algebra $\mathcal{O}_N$ with $2 \leqslant N < \infty$ [6] and real numbers.
(iii) From Theorem 1.6, we suspect that there exists a relation between the group of all modular transformations and $\mathcal{O}_\infty$. Make clear this relation.

In Section 2, we prove Theorems 1.4 and 1.6. In Section 3, we show examples of Theorem 1.6.

2. Proofs of the theorems

In this section, we prove the main theorems.

2.1. Continued fraction expansion and one-sided full shift

We review relations between continued fraction expansions and the one-sided full shift on $\mathbf{N}^\infty$ in this subsection. Let $[a_1(x), a_2(x), \ldots]$ denote (1.2) for simplicity of description.

Fact 2.1. (See [9, Theorem 175].) For the equivalence in Definition 1.5, two irrational numbers x and y are equivalent if and only if there exist positive integers $a_1, \ldots, a_m$, $b_1, \ldots, b_n$ such that

$$x = [a_1, \ldots, a_m, c_1, c_2, \ldots], \qquad y = [b_1, \ldots, b_n, c_1, c_2, \ldots]. \tag{2.1}$$

Define the *simple continued fraction transformation* (*or the Gauss map*) τ from Ω to Ω by

$$\tau(x) \equiv \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \quad (x \in \Omega), \tag{2.2}$$

where $\lfloor \cdot \rfloor$ denotes the floor (entire) function [10,17]. Then we see that $\tau([a_1, a_2, \ldots]) = [a_2, a_3, \ldots]$ for the continued fraction $[a_1, a_2, \ldots]$.

Define the map σ from $\mathbf{N}^\infty$ to $\mathbf{N}^\infty$ by

$$\sigma(n_1, n_2, \ldots) \equiv (n_2, n_3, \ldots). \tag{2.3}$$

The dynamical system $(\mathbf{N}^\infty, \sigma)$ is called the *one-sided full shift* on $\mathbf{N}^\infty$ [15, §7.2]. Then we see that CFE in (1.3) satisfies the following equation [10, (1.1.2)]:

$$\mathsf{CFE}\big(\tau(x)\big) = \sigma\big(\mathsf{CFE}(x)\big) \quad (x \in \Omega). \tag{2.4}$$

From (2.4), two dynamical systems (Ω, τ) and $(\mathbf{N}^\infty, \sigma)$ are conjugate.

Definition 2.2. For $a, b \in \mathbf{N}^\infty$, let $a \sim b$ denote when there exist $p, q \geqslant 1$ such that $\sigma^p(a) = \sigma^q(b)$.

We call $\sim$ the *tail equivalence* in $\mathbf{N}^\infty$ [2, Chapter 2]. Then the following holds by definition.

Fact 2.3. For $x, y \in \Omega$, $x \sim y$ if and only if $\mathsf{CFE}(x) \sim \mathsf{CFE}(y)$.

For $\{\alpha_i\colon\ i\in\mathbf{N}\}$ and $\{\beta_i\colon\ i\in\mathbf{N}\}$ in Definition 1.3, the following holds:

$$\mathsf{CFE}\big(\alpha_i(x)\big)=\beta_i\big(\mathsf{CFE}(x)\big)\quad(x\in\Omega,\ i\in\mathbf{N}).\tag{2.5}$$

Remark that they are closely related to τ and σ as follows:

$$(\tau\circ\alpha_i)(x)=x,\qquad(\sigma\circ\beta_i)(a)=a\quad\big(x\in\Omega,\ a\in\mathbf{N}^\infty,\ i\in\mathbf{N}\big).\tag{2.6}$$

2.2. Properties of representations of $\mathcal{O}_\infty$

In this subsection, we recall properties of permutative representations and the shift representation of $\mathcal{O}_\infty$. Define $\mathbf{N}^*\equiv\bigcup_{1\leqslant k<\infty}\mathbf{N}^k$. For $J=(j_l)_{l=1}^m, K=(k_l)_{l=1}^{m'}\in\mathbf{N}^*$, we write $J\sim K$ if $m=m'$ and $K=pJ$ where $pJ=(j_{p(1)},\dots,j_{p(m)})$ for any cyclic permutation $p\in\mathbf{Z}_m$. For $J\in\mathbf{N}^*$, we call J *nonperiodic* if $pJ\neq J$ for any cyclic permutation $p\neq id$. If $J\in\mathbf{N}^*$ is nonperiodic, then we see that $(\mathcal{H},\pi)$ is $P(J)$ if and only if there exists a cyclic vector $\mathrm{v}\in\mathcal{H}$ such that $\pi(s_J)\mathrm{v}=\mathrm{v}$. For $J\in\mathbf{N}^\infty$, we call J *nonperiodic* if J has no repeating block.

Proposition 2.4. *Let $P(J)$ be as in Definition* 1.2.

(i) *Any permutative representation of $\mathcal{O}_\infty$ is decomposed into the direct sum of cyclic permutative representations uniquely up to unitary equivalence.*
(ii) *Any cyclic permutative representation is either one of the following two cases*:
  (a) *$P(J)$ for $J\in\mathbf{N}^*$.*
  (b) *$P(J)$ for $J\in\mathbf{N}^\infty$.*
(iii) *If $J\in\mathbf{N}^*$ and $K\in\mathbf{N}^\infty$, then $P(J)\neq P(K)$.*
(iv) *For $J,K\in\mathbf{N}^*\cup\mathbf{N}^\infty$, $P(J)=P(K)$ if and only if $J\sim K$ where we define $J\nsim K$ when $J\in\mathbf{N}^*$ and $K\in\mathbf{N}^\infty$.*
(v) *For $J\in\mathbf{N}^*\cup\mathbf{N}^\infty$, $P(J)$ is irreducible if and only if J is nonperiodic.*

Proof. See Appendix A.1. □

Next, we show properties of the shift representation of $\mathcal{O}_\infty$ (see also [2, Chapter 6]).

Proposition 2.5. *Let $(l_2(\mathbf{N}^\infty),\pi_\beta)$ be as in Definition* 1.3(ii). *Define $[a]\equiv\{b\in\mathbf{N}^\infty\colon\ b\sim a\}$ where $\sim$ is as in Definition* 2.2.

(i) *The following irreducible decomposition holds*:

$$l_2\big(\mathbf{N}^\infty\big)=\bigoplus_{[a]\in\mathbf{N}^\infty/\sim}\mathcal{K}_{[a]}\tag{2.7}$$

where $\mathcal{K}_{[a]}$ denotes the closed subspace of $l_2(\mathbf{N}^\infty)$ generated by the subset $\{e_b\colon\ b\in[a]\}$ of $\{e_c\colon\ c\in\mathbf{N}^\infty\}$.
(ii) *For $a\in\mathbf{N}^\infty$, let $\theta_{[a]}$ denote the subrepresentation of π_β associated with the subspace $\mathcal{K}_{[a]}$, that is,*

$$\theta_{[a]}\equiv\pi_\beta|_{\mathcal{K}_{[a]}}.\tag{2.8}$$

Then $\theta_{[a]}$ and $\theta_{[b]}$ are unitarily equivalent if and only if $a\sim b$. Especially, (2.7) *is multiplicity free.*
(iii) (a) *If $a\in\mathbf{N}^\infty$ has no repeating block, then $\theta_{[a]}$ is $P(a)$.*
  (b) *If $a\in\mathbf{N}^\infty$ has the repeating block a', then $\theta_{[a]}$ is $P(a')$.*
(iv) *Any irreducible permutative representation of $\mathcal{O}_\infty$ is unitarily equivalent to $\theta_{[a]}$ for some $[a]\in\mathbf{N}^\infty/\sim$.*

Proof. See Appendix A.2. □

2.3. Proofs of Theorems 1.4 and 1.6

We prove Theorems 1.4 and 1.6 in this subsection.

Proof of Theorem 1.4. Let CFE be as in (1.3). Define the unitary U from $l_2(\Omega)$ to $l_2(\mathbf{N}^\infty)$ by

$$Ue_x \equiv e'_{\mathrm{CFE}(x)} \quad (x \in \Omega) \tag{2.9}$$

where we write $\{e_x\colon\ x \in \Omega\}$ and $\{e'_a\colon\ a \in \mathbf{N}^\infty\}$ as standard basis of $l_2(\Omega)$ and $l_2(\mathbf{N}^\infty)$, respectively. From (2.5), we can verify that $U\pi_\alpha(s_i)U^* = \pi_\beta(s_i)$ for each $i \in \mathbf{N}$. This implies the unitary equivalence between two representations $(l_2(\Omega), \pi_\alpha)$ and $(l_2(\mathbf{N}^\infty), \pi_\beta)$:

$$U\pi_\alpha(A)U^* = \pi_\beta(A) \quad (A \in \mathcal{O}_\infty). \tag{2.10}$$

Hence the statement holds. □

Proof of Theorem 1.6. From Fact 2.3, we see that

$$\mathrm{CFE}\big([x]\big) = \big[\mathrm{CFE}(x)\big] \quad (x \in \Omega). \tag{2.11}$$

By this and (2.9),

$$U\mathcal{H}_{[x]} = \mathcal{K}_{[\mathrm{CFE}(x)]} \quad (x \in \Omega). \tag{2.12}$$

From Proposition 2.5(i) and Theorem 1.4, the statement of (i) holds.

By (2.10) and (2.12), we obtain that

$$U\eta_{[x]}(A)U^* = \theta_{[\mathrm{CFE}(x)]}(A) \quad (x \in \Omega,\ A \in \mathcal{O}_\infty). \tag{2.13}$$

From Proposition 2.5(ii), (iii), (iv) and (2.13), the statements of (ii), (iii) and (iv) hold, respectively. □

3. Permutative representations of $\mathcal{O}_\infty$ associated with quadratic irrationals

In order to take a closer look at the correspondence between representations and irrationals in Theorem 1.6, it is better to check concrete representations and irrationals by computation. We show examples of Theorem 1.6(iii)(b) in this section. In Theorem 1.6(iii)(b), a representations is characterized by a finite sequence of natural numbers, and an irrational is a quadratic irrational. Hence both representations and irrationals are easier to treat than cases in Theorem 1.6(iii)(a). For example, the (class of) representation $P(1)$ is characterized only by a cyclic unit vector v which satisfies the following equation:

$$s_1\mathrm{v} = \mathrm{v}. \tag{3.1}$$

The eigenequation of the operator s_1 in (3.1) is interpreted as the following algebraic equation according to (1.6) and (1.7):

$$\frac{1}{x+1} = \alpha_1(x) = x. \tag{3.2}$$

Of course, the solution of (3.2) in $0 < x < 1$ is the inverse $(\sqrt{5}-1)/2$ of the golden ratio $(\sqrt{5}+1)/2$. In this way, this and similar cases are suitable for an illustration of Theorem 1.6 and for the computation of the corresponded irrational number.

Example 3.1. For $k \in \mathbf{N}$, define $x \in \Omega$ by

$$x = \frac{\sqrt{k^2+4}-k}{2}. \tag{3.3}$$

Then x is equal to $[k, k, k, \ldots]$ as an infinite continued fraction. Hence the repeating block $\mathsf{CFE}_0(x)$ of x is k. Therefore $\eta_{[x]}$ in (1.12) is $P(k)$ from Theorem 1.6(iii)(b). For example,

$$\eta_{[\frac{\sqrt{5}-1}{2}]} = P(1), \qquad \eta_{[\sqrt{2}-1]} = P(2), \qquad \eta_{[\frac{\sqrt{13}-3}{2}]} = P(3). \tag{3.4}$$

Let $\mathcal{B}$ denote the $*$-algebra generated by $\{b_n\colon\ n \geqslant 1\}$ which satisfies

$$b_n b_m^* - b_m^* b_n = \delta_{nm} I, \qquad b_n b_m - b_m b_n = 0 \quad (n, m \geqslant 1). \tag{3.5}$$

The algebra $\mathcal{B}$ is called the *algebra of bosons*. In [13], we showed that b_n's are written by using canonical generators of $\mathcal{O}_\infty$ on any permutative representation of $\mathcal{O}_\infty$. From this, the restriction of $P(1)$ on $\mathcal{B}$ is unitarily equivalent to the Fock representation of $\mathcal{B}$ [13, Theorem 1.1(i)]. Especially, the following holds:

$$b_n \mathrm{v} = 0 \quad (n \geqslant 1), \tag{3.6}$$

that is, the vector v in (3.1) is the *vacuum* of the Fock representation of $\mathcal{B}$. Hence the first equation in (3.4) shows a relation between the inverse $(\sqrt{5}-1)/2$ of the golden ratio and the Fock representation of bosons.

Example 3.2. For $j, k \in \mathbf{N}$, define $x \in \Omega$ by

$$x = \frac{\sqrt{(jk)^2+4jk}-jk}{2j}. \tag{3.7}$$

When $j \neq k$,

$$\eta_{[x]} = P(j, k). \tag{3.8}$$

For example, $\eta_{[\sqrt{3}-1]} = P(1, 2)$. If $j = k$, then (3.7) equals to (3.3). Therefore $\eta_{[x]}$ is $P(k)$.

Example 3.3. For $i, j, k \in \mathbf{N}$, define $x \in \Omega$ by

$$x = \frac{-(ijk+i+k-j)+\sqrt{D}}{2(ij+1)}, \tag{3.9}$$

$$D = (ijk+i+j+k)^2+4. \tag{3.10}$$

If (i, j, k) is nonperiodic, then $\eta_{[x]} = P(i, j, k)$. If $i = j = k$, then (3.9) equals to (3.3).

Example 3.4. For $i, j, k, l \in \mathbf{N}$, define $x \in \Omega$ by

$$x = \frac{-(ijkl+ij+kl+li-jk)+\sqrt{D}}{2(ijk+i+k)}, \tag{3.11}$$

$$D = (ijkl+ij+jk+kl+li)(ijkl+ij+jk+kl+li+4). \tag{3.12}$$

If (i, j, k, l) is nonperiodic, then $\eta_{[x]} = P(i, j, k, l)$.

Remark that the symbol D in neither (3.10) nor (3.12) always means the discriminant of x. For example, when $(i,j,k)=(1,2,3)$, D in (3.10) is 148. On the other hand, x in (3.9) is $\frac{-5+\sqrt{37}}{2}$ with the discriminant 37.

Problem 3.5. For a given $J\in\mathbf{N}^n$ for $n\geqslant 5$, compute $x\in\Omega$ such that $\eta_{[x]}=P(J)$.

Acknowledgments

The authors would like to express their sincere thanks to the referees for their numerous suggestions.

Appendix A. Proofs of the propositions

We prove Propositions 2.4 and 2.5 in this section.

A.1. Proof of Proposition 2.4

Let $(\mathcal{H},\pi)$ be a permutative representation of $\mathcal{O}_\infty$. By assumption, we see that there exist a family $\{f_i\colon\ i\in\mathbf{N}\}$ of maps on a set Λ and an orthonormal basis $\{e_n\colon\ n\in\Lambda\}$ such that

$$\pi(s_i)e_n=e_{f_i(n)}\quad(i\in\mathbf{N},\ n\in\Lambda).\tag{A.1}$$

From this and (1.5), we see that $\#\Lambda=\infty$, the map $f_i:\Lambda\to\Lambda$ is injective for each i, $f_i(\Lambda)\cap f_j(\Lambda)=\emptyset$ when $i\neq j$ and $\Lambda=\bigcup_{i\in\mathbf{N}}f_i(\Lambda)$. The family $\{f_i\colon\ i\in\mathbf{N}\}$ is called a *branching function system* [2, Definition 2.1]. On the other hand, if a branching function system is given, then we can construct a permutative representation of $\mathcal{O}_\infty$ as (A.1). Hence we write π in (A.1) as π_f. For a given branching function system $\{f_i\colon\ i\in\mathbf{N}\}$, define the *coding map* F of $\{f_i\colon\ i\in\mathbf{N}\}$ by the map from Λ to Λ as $F(n)\equiv f_i^{-1}(n)$ when $n\in f_i(\Lambda)$.

Proof of Proposition 2.4. (i) Let $(\mathcal{H},\pi)$ be a permutative representation of $\mathcal{O}_\infty$. Then there exists a branching function system $f=\{f_i\colon\ i\in\mathbf{N}\}$ on a set X such that $(\mathcal{H},\pi)$ is unitarily equivalent to $(l_2(X),\pi_f)$ by definition. For $x,y\in X$, define the equivalence relation $\sim$ as $x\sim y$ if and only if there exist $i,j\in\mathbf{N}$ such that $F^i(x)=F^j(y)$ where F denotes the coding map of f. Let $\{X_\lambda\colon\ \lambda\in\Xi\}$ denote the set of all equivalence classes in X with respect to $\sim$. Then we see that X is decomposed into the disjoint union of subsets $\{X_\lambda\colon\ \lambda\in\Xi\}$ such that $f_i(X_\lambda)\subset X_\lambda$ for each $i\in\mathbf{N}$ and $\lambda\in\Xi$. Therefore f is decomposed into the direct sum of branching function systems $f^{(\lambda)}\equiv\{f_i|_{X_\lambda}\colon\ i\in\mathbf{N}\}$ for $\lambda\in\Xi$. By the definition of X_λ, $(l_2(X_\lambda),\pi_{f^{(\lambda)}})$ is cyclic and $\pi_{f^{(\lambda)}}=\pi_f|_{l_2(X_\lambda)}$. This implies that $\mathcal{H}$ is decomposed into the direct sum of cyclic subspaces $\{l_2(X_\lambda)\colon\ \lambda\in\Xi\}$. Hence the statement of decomposition holds.

Assume that $g=\{g_i\colon\ i\in\mathbf{N}\}$ is another branching function system on a set Y associated with π. Since dimensions of the representation space of both π_f and π_g are same, we can assume that $Y=X$. Define the map ϕ from X to X by $\phi(x)\equiv(g_i\circ f_i^{-1})(x)$ when $x\in f_i(X)$. Then ϕ is bijective and induces the same decomposition up to conjugacy. Define the unitary U on $\mathcal{H}$ by $Ue_x\equiv e_{\phi(x)}$ for the standard basis $\{e_x\colon\ x\in X\}$. Then we see that $U\pi_f(\cdot)U^*=\pi_g$. Hence the statement of uniqueness holds.

(ii) Let $(\mathcal{H},\pi)$ be a cyclic permutative representation of $\mathcal{O}_\infty$. From the proof of (i), we can assume that there exists a branching function system $f=\{f_i\colon\ i\in\mathbf{N}\}$ on a set X such that $(\mathcal{H},\pi)$ is unitarily equivalent to $(l_2(X),\pi_f)$ with a cyclic vector e_{x_0} for a point $x_0\in X$. For the coding map F of f, define the sequence $x_n\equiv F^n(x_0)$ for $n\geqslant 0$. If there exist $m_0,k\geqslant 1$ such that $x_{m+k}=x_m$ for each $m\geqslant m_0$, then we see that $(\mathcal{H},\pi)$ is the case of (a). If not, then $(\mathcal{H},\pi)$ is the case of (b).

(iii) If they are equivalent, then there exists an action of $\mathcal{O}_\infty$ on a Hilbert space $\mathcal{H}$ with two cyclic unit vectors v and v' such that v and v' are GP vectors of $P(J)$ and $P(K)$, respectively. Then $\langle\mathrm{v}|\mathrm{v}'\rangle=\langle\mathrm{v}|s_J^*\mathrm{v}'\rangle=\cdots=\langle\mathrm{v}|(s_J^*)^n\mathrm{v}'\rangle\to 0$ $(n\to\infty)$ because of the assumption of $P(K)$. From this, $\langle\mathrm{v}|\mathrm{v}'\rangle=0$. This implies that $\langle s_L\mathrm{v}|\mathrm{v}'\rangle=0$ for each $L\in\mathbf{N}^*$. Therefore $\mathrm{v}'=0$. This is a contradiction.

(iv) From (iii), it is sufficient to show the following two cases.

(a) Assume $J, K \in \mathbf{N}^*$. We can assume that $\mathcal{O}_\infty$ acts on two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ with cyclic unit vectors v, v' which are GP vectors of $P(J)$ and $P(K)$, respectively. If $J \sim K$, then we can find $\mathrm{v}'' \in \mathcal{H}'$ such that $s_J \mathrm{v}'' = \mathrm{v}''$ and $\{s_{j_k} \cdots s_{j_n} \mathrm{v}'' \colon\ k = 1, \ldots, n\}$ is an orthonormal family when $J = (j_1, \ldots, j_n)$. Define the unitary U from $\mathcal{H}$ to $\mathcal{H}'$ by $U s_L \mathrm{v} = s_L \mathrm{v}''$ for each $L \in \mathbf{N}^*$. Then U gives the unitary equivalence of $\mathcal{H}$ and $\mathcal{H}'$.
Assume that $P(J) = P(K)$, but $J \nsim K$. Then we can assume that $\mathcal{O}_\infty$ acts on a Hilbert space $\mathcal{H}$ with two cyclic unit vectors v and v' which are GP vectors of $P(J)$ and $P(K)$, respectively. Then $\langle \mathrm{v} | \mathrm{v}' \rangle = \langle \mathrm{v} | (s_J^*)^n s_K^m \mathrm{v}' \rangle$ for each $n, m \geqslant 1$. Since $J \nsim K$, we see that $(s_J^*)^n s_K^m = 0$ for some n, m. Therefore $\langle \mathrm{v} | \mathrm{v}' \rangle = 0$. This implies that $\langle s_L \mathrm{v} | \mathrm{v}' \rangle = 0$ for each $L \in \mathbf{N}^*$. Therefore $\mathrm{v}' = 0$. This is a contradiction.
(b) Assume $J, K \in \mathbf{N}^\infty$. Then $P(J) = P(K)$ if and only if $J \sim K$ from the analogy of the case (a).

(v) We consider the following two cases.

(a) Assume $J \in \mathbf{N}^*$. Assume that J is nonperiodic and $\mathcal{O}_\infty$ acts on a Hilbert space $\mathcal{H}$ with a cyclic unit vector v which is the GP vector of $P(J)$. Let $w \in \mathcal{H}$. Since $\mathcal{H}$ is generated by the set $\{s_K \mathrm{v} \colon\ K \in \mathbf{N}^*\}$, we can write $w = \sum_L a_L s_L \mathrm{v}$ such that the set $\{s_L \mathrm{v}\}$ is an orthonormal family in $\mathcal{H}$. If $w \neq 0$, there exists $L_0 \in \mathbf{N}^*$ such that $a_{L_0} \neq 0$. By replacing w by $a_{L_0}^{-1} s_{L_0}^* w$, we can always assume $\langle w | \mathrm{v} \rangle = 1$. On the other hand, $s_J^* s_K \mathrm{v} = 0$ when $K \neq J^n$ for some $n \geqslant 0$. Hence $(s_J^*)^n w \to \mathrm{v}$ when $n \to \infty$. From this, we can obtain v from a given $w \in \mathcal{H}$, $w \neq 0$. This implies that $\mathcal{H}$ is irreducible.
Assume that J is not nonperiodic (= periodic). Then there exists $J_0 \in \mathbf{N}^*$ such that J is the n-times concatenation of J_0 for some $n \geqslant 2$. Assume that a unit vector v satisfies $s_J \mathrm{v} = \mathrm{v}$. Define two vectors w_1 and w_2 by

$$w_1 \equiv \mathrm{v} + s_{J_0} \mathrm{v} + \cdots + s_{J_0}^{n-1} \mathrm{v}, \tag{A.2}$$

$$w_2 \equiv \mathrm{v} + \zeta s_{J_0} \mathrm{v} + \cdots + \zeta^{n-1} s_{J_0}^{n-1} \mathrm{v} \tag{A.3}$$

where $\zeta \equiv e^{2\pi\sqrt{-1}/n}$. Since $s_{J_0} w_1 = w_1$ and $s_{J_0} w_2 = \zeta^{-1} w_2$, $\langle w_1 | w_2 \rangle = 0$. Define $V_i \equiv \overline{\mathcal{O}_\infty w_i}$ for $i = 1, 2$. Then we can verify that V_1 and V_2 are orthogonal. Since $\{0\} \neq V_i \subsetneq V_1 \oplus V_2 \subset \mathcal{H}$ for $i = 1, 2$, $\mathcal{H}$ is not irreducible.
(b) Assume $J \in \mathbf{N}^\infty$. If J is nonperiodic, then we can prove the irreducibility of $P(J)$ by the analogy of the case of (a).
Assume that J is not nonperiodic. We can assume that there exists $J_0 \in \mathbf{N}^*$ such that J is purely periodic with the repeating block J_0. Assume that $\mathcal{O}_\infty$ acts on a Hilbert space $\mathcal{K}$ with a cyclic unit vector v_0 such that $s_{J_0} \mathrm{v}_0 = \mathrm{v}_0$. Let $\mathcal{L}$ denote the Hilbert space of all $\mathcal{K}$-valued functions ϕ on $U(1) \equiv \{z \in \mathbf{C} \colon\ |z| = 1\}$ such that $\int_{U(1)} \|\phi(z)\|^2 \, d\mu(z) < \infty$ where μ denotes the probabilistic Haar measure of the unitary group $U(1)$. Define the action of $\mathcal{O}_\infty$ on $\mathcal{L}$ by

$$(s_i \phi)(z) \equiv z s_i \phi(z) \quad \big(z \in U(1),\ \phi \in \mathcal{L},\ i \in \mathbf{N}\big). \tag{A.4}$$

Let $\tilde{\mathrm{v}}_0$ denote the constant function in $\mathcal{L}$ such that $\tilde{\mathrm{v}}_0(z) \equiv \mathrm{v}_0$ for each $z \in U(1)$. Then $(s_{J_0} \tilde{\mathrm{v}}_0)(z) = z \tilde{\mathrm{v}}_0(z)$ for each $z \in U(1)$. From this, we see that $\{(s_{J_{(n)}})^* \tilde{\mathrm{v}}_0 \colon\ n \geqslant 1\}$ is an orthonormal family in $\mathcal{L}$. Let $\mathcal{L}_1$ denote the cyclic subspace of $\mathcal{L}_1$ generated by $\tilde{\mathrm{v}}_0$ with respect to the action of $\mathcal{O}_\infty$. Then $\mathcal{L}_1$ is $P(J)$. Define the operator T on $\mathcal{L}$ by $(T\phi)(z) \equiv z\phi(z)$ for $\phi \in \mathcal{L}$ and $z \in U(1)$. Then $T\mathcal{L}_1 \subset \mathcal{L}_1$ and $T s_i = s_i T$ and $T s_i^* = s_i^* T$ for each $i \in \mathbf{N}$ on $\mathcal{L}_1$. Hence T is the nontrivial element of the commutant of $\mathcal{O}_\infty$ on $\mathcal{L}_1$. Therefore neither $\mathcal{L}_1$ nor $\mathcal{L}$ is irreducible. Hence $P(J)$ is not irreducible. □

A.2. Proof of Proposition 2.5

(i) For $[a] \in \mathbf{N}^\infty/\sim$, we see that $\beta_i([a]) \subset [a]$ for each i. This implies that $\pi_\beta(\mathcal{O}_\infty)\mathcal{K}_{[a]} \subset \mathcal{K}_{[a]}$. By definition, $l_2(\mathbf{N}^\infty)$ is decomposed into the direct sum of the family $\{\mathcal{K}_{[a]}\colon\ [a] \in \mathbf{N}^\infty/\sim\}$ of closed subspaces as a Hilbert space, the statement of decomposition holds. It is sufficient to show the irreducibility of $\mathcal{K}_{[a]}$ for each $[a]$. We consider the following two cases.

(a) Assume that a has the repeating block a'. Let $(a')^\infty$ denote the purely periodic sequence with the repeating block a'. Then $[a] = \{\beta_J(a')\colon\ J \in \mathbf{N}^k,\ k \geqslant 0\}$ and $\beta_{a'}((a')^\infty) = (a')^\infty$ where $\beta_J \equiv \beta_{j_1} \circ \cdots \circ \beta_{j_k}$ when $J = (j_1, \ldots, j_k)$. This implies that $(\mathcal{K}_{[a]}, \pi_\beta|_{\mathcal{K}_{[a]}})$ is $P(a')$ with the GP vector $e_{(a')^\infty}$. From Proposition 2.4(v) and the nonperiodicity of a', $\mathcal{K}_{[a]}$ is irreducible.
(b) Assume that a has no repeating block. By the analogy of the case of (a), we see that $\mathcal{K}_{[a]}$ is $P(a)$ with the GP vector e_a. From Proposition 2.4(v) and the nonperiodicity of a, $\mathcal{K}_{[a]}$ is irreducible.

(ii) From the proof of (i), $\theta_{[a]}$ is $P(J)$ for a nonperiodic element J in $\mathbf{N}^* \cup \mathbf{N}^\infty$. From this and Proposition 2.4(iii) and (iv), the statement holds.

(iii) The proof has been already given in the proof of (i).

(iv) Let $(\mathcal{H}, \pi)$ be an irreducible permutative representation of $\mathcal{O}_\infty$. From Proposition 2.4(ii), $(\mathcal{H}, \pi)$ is $P(J)$ for some nonperiodic element J in $\mathbf{N}^* \cup \mathbf{N}^\infty$. From (iii), if $J \in \mathbf{N}^*$, then $\theta_{[a]}$ is $P(J)$ for $a \equiv J^\infty \in \mathbf{N}^\infty$, and if $J \in \mathbf{N}^\infty$, then $\theta_{[J]}$ is $P(J)$. Hence the statement holds.

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Journal of Number Theory 133 (2013) 2153–2181

Contents lists available at SciVerse ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

A Gauss–Kuzmin-type problem for a family of continued fraction expansions

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ARTICLE INFO

Article history:
Received 27 September 2011
Revised 17 December 2012
Accepted 17 December 2012
Available online 5 March 2013
Communicated by David Goss

Keywords:
Invariant measure
Perron–Frobenius operator
Random system with complete connections

ABSTRACT

In this paper we study in detail a family of continued fraction expansions of any number in the unit closed interval $[0, 1]$ whose digits are differences of consecutive non-positive integer powers of an integer $m \geqslant 2$. For the transformation which generates this expansion and its invariant measure, the Perron–Frobenius operator is given and studied. For this expansion, we apply the method of random systems with complete connections by Iosifescu and obtained the solution of its Gauss–Kuzmin type problem.

1. Introduction

The purpose of this paper is to prove a Gauss–Kuzmin type problem for non-regular continued fraction expansions introduced by Chan [5]. In order to solve the problem, we apply the random systems with complete connections by Iosifescu [10]. First we outline the historical framework of this problem. Then, in Section 1.2, we present the current framework. The main theorem will be shown in Section 1.3. In this subsection we will also give a detailed outline of the paper.

1.1. Gauss' Problem

One of the first and still one of the most important results in the metrical theory of continued fractions is so-called Gauss–Kuzmin theorem. Write $x \in [0, 1)$ as a regular continued fraction

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0022-314X/$ – see front matter
http://dx.doi.org/10.1016/j.jnt.2012.12.007

$$x=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\ddots}}}:=[a_1,a_2,a_3,\ldots],$$

where $a_n\in\mathbb{N}_+:=\{1,2,3,\ldots\}$. The metrical theory of continued fractions started on 25th October 1800, with a note by Gauss in his mathematical diary. Gauss wrote that (in modern notation)

$$\lim_{n\to\infty}\lambda\big(\tau^n\leqslant x\big)=\frac{\log(1+x)}{\log 2},\quad x\in I:=[0,1].$$

Here λ is a Lebesgue measure and the map $\tau:[0,1)\to[0,1)$, the so-called *regular continued fraction* (or *Gauss*) transformation, is defined by

$$\tau(x):=\frac{1}{x}-\left\lfloor\frac{1}{x}\right\rfloor,\quad x\neq 0;\qquad \tau(0):=0,$$

where $\lfloor\cdot\rfloor$ denotes the *floor* (or *entire*) function. Gauss' proof (if any) has never been found. A little more than 11 years later, in a letter dated 30 January 1812, Gauss asked Laplace to estimate the error

$$e_n(x):=\lambda\big(\tau^{-n}[0,x]\big)-\frac{\log(1+x)}{\log 2},\quad n\geqslant 1,\ x\in I.$$

This has been called *Gauss' Problem*. It received a first solution more than a century later, when R.O. Kuzmin (see [16]) showed in 1928 that $e_n(x)=\mathcal{O}(q^{\sqrt{n}})$ as $n\to\infty$, uniformly in x with some (unspecified) $0<q<1$. One year later, using a different method, Paul Lévy (see [17]) improved Kuzmin's result by showing that $|e_n(x)|\leqslant q^n$, $n\in\mathbb{N}_+$, $x\in I$, with $q=3.5-2\sqrt{2}=0.67157\ldots$. The Gauss–Kuzmin–Lévy theorem is the first basic result in the rich metrical theory of continued fractions.

1.2. A non-regular continued fraction expansion

In this paper, we consider a generalization of the Gauss transformation and prove an analogous result.

In [5], Chan shows that any $x\in[0,1)$ can be written in the form

$$x=\cfrac{m^{-a_1(x)}}{1+\cfrac{(m-1)m^{-a_2(x)}}{1+\cfrac{(m-1)m^{-a_3(x)}}{1+\ddots}}}:=\big[a_1(x),a_2(x),a_3(x),\ldots\big]_m,\tag{1.1}$$

where $m\in\mathbb{N}_+$, $m\geqslant 2$ and $a_n(x)$s are non-negative integers.

For any $m\in\mathbb{N}_+$ with $m\geqslant 2$, define the transformation τ_m on I by

$$\tau_m(x)=\begin{cases}\dfrac{m^{\{\frac{\log x^{-1}}{\log m}\}}-1}{m-1}, & \text{if } x\neq 0,\\ 0, & \text{if } x=0,\end{cases}\tag{1.2}$$

where $\{\cdot\}$ stands for fractionary part. It is easy to see that τ_m maps the set Ω of irrationals in I into itself. For any $x\in(0,1)$ put

$$a_n = a_n(x) = a_1\big(\tau_m^{n-1}(x)\big), \quad n \in \mathbb{N}_+, \tag{1.3}$$

with $\tau_m^0(x) = x$ and

$$a_1 = a_1(x) = \begin{cases} \lfloor \log x^{-1} / \log m \rfloor, & \text{if } x \neq 0, \\ \infty, & \text{if } x = 0. \end{cases} \tag{1.4}$$

Transformation τ_m which generates the continued fraction expansion (1.1) is ergodic with respect to an invariant probability measure, γ_m, where

$$\gamma_m(A) = k_m \int_A \frac{dx}{((m-1)x+1)((m-1)x+m)}, \quad A \in \mathcal{B}_I,$$

with $k_m = \frac{(m-1)^2}{\log(m^2/(2m-1))}$ and $\mathcal{B}_I$ is the σ-algebra of Borel subsets of I (which, by definition, is the smallest σ-algebra containing intervals).

The ergodicity of τ_m plays a key role in the study of the asymptotic growth rate of the random Fibonacci type sequences $\{f_n\}$ defined by $f_{-1} = 0$, $f_0 = 1$, $c_0 = 0$ and

$$f_n = m^{c_n} f_{n-1} + (m-1) m^{c_{n-1}} f_{n-2}, \tag{1.5}$$

where c_n, $n \geqslant 1$, are the digits from (1.1). As is known, the Fibonacci sequence is defined using the linear recurrence relation

$$F_{n+1} = F_n + F_{n-1}, \quad n \in \mathbb{N}_+, \quad \text{with } F_0 = F_1 = 1,$$

and Binet's formula is

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}, \quad n \in \mathbb{N}.$$

It is known that using Binet's formula we can compute the asymptotic growth rate of the Fibonacci sequence $\{F_n\}$, which is given by

$$\lim_{n\to\infty} \frac{1}{n} \log F_n = \log\left(\frac{1+\sqrt{5}}{2}\right) = 0.4812\ldots.$$

In the case of random Fibonacci type sequences, defined by (with fixed f_1 and f_2)

$$f_n = \alpha(n) f_{n-1} + \beta(n) f_{n-2},$$

where $\alpha(n)$ and $\beta(n)$ are random coefficients, the quest for the asymptotic growth rate is more difficult. Recently, Viswanath (see [25]) proved that the asymptotic growth rate of the random Fibonacci sequences defined by $f_1 = f_2 = 1$ and

$$f_n = \pm f_{n-1} \pm f_{n-2},$$

where the signs are chosen independently and with equal probabilities, is given by

$$\lim_{n\to\infty} \frac{1}{n} \log f_n = \log(1.13198824\ldots) = 0.12397559\ldots$$

with probability 1. But Viswanath's method is not the only way through. So, Chan proved in [5] that for almost all x with respect to the Lebesgue measure, the asymptotic growth rate of $\{f_n\}$ from (1.5) is given by

$$\lim_{n\to\infty}\frac{1}{n}\log f_n = k_m\int_0^1 \frac{\log(1/x)}{((m-1)x+1)((m-1)x+m)}\,dx$$
$$\leqslant k_m\frac{3m-1}{2m(2m-1)}. \qquad \square$$

1.3. Main theorem

We show our main theorem in this subsection. For this purpose let μ be a non-atomic probability measure on $\mathcal{B}_I$ and define

$$F_n(x) = \mu\big(\tau_m^n < x\big), \quad x\in I,\ n\in\mathbb{N},$$
$$F(x) = \lim_{n\to\infty} F_n(x), \quad x\in I,$$

with $F_0(x) = \mu([0,x))$.

Then the following holds.

Theorem 1.1 *(A Gauss–Kuzmin-type theorem). If μ has a Riemann-integrable density, then*

$$F(x) = \frac{k_m}{(m-1)^2}\log\frac{m((m-1)x+1)}{(m-1)x+m}, \quad x\in I, \tag{1.6}$$

where $k_m = \frac{(m-1)^2}{\log(m^2/(2m-1))}$.

If the density of μ is a Lipschitz function, then there exist two positive constants $q<1$ and k such that for all $x\in I$ and $n\in\mathbb{N}_+$ we have

$$\mu\big(\tau_m^n < x\big) = \frac{k_m}{(m-1)^2}\big(1+\theta q^n\big)\log\frac{m((m-1)x+1)}{(m-1)x+m}, \tag{1.7}$$

where θ is a certain constant determined by μ, n, x such that $|\theta|\leqslant k$.

The paper is organized as follows. In Section 2 we give the basic metric properties of the continued fraction expansion in (1.1). Hence, we give a Legendre-type result and the Brodén–Borel–Lévy formula used to determine the probability structure of $(a_n)_{n\in\mathbb{N}_+}$ under λ. In Section 2.4, we find the invariant measure of τ_m. The proof of this result is given in a different manner from that described by Chan in [5]. In Section 3 we consider the so-called *natural extension* $\overline{\tau}_m$ (see [19]), define extended incomplete quotients $\bar{a}_l$, $l\in\mathbb{Z}$, and we generalize some results presented in Section 2. In Section 4, we derive the associated Perron–Frobenius operator under different probability measures on $\mathcal{B}_I$. We study the Perron–Frobenius operator of τ_m under the invariant measure γ_m induced by the limit distribution function, we derive the asymptotic behavior of this operator and we restrict the Perron–Frobenius operator to the linear space of all complex-valued functions of bounded variation and to the space of all bounded measurable complex-valued functions. Section 5 is divided into three parts. The first subsection has as purpose defining the notion of random system with complete connections. In the second subsection we set up the necessary machinery to prove the main theorem whose proof is contained in the last subsection. To determine where $\mu(\tau_m^n < x)$ tends as $n\to\infty$ and give the rate of this convergence, we use the ergodic behavior of the random system with complete connections

associated with this expansion. For a more detailed study of the theory and applications of dependence with complete connections to the metrical problems and other interesting aspects of number theory we refer the reader to [10,12,13,22–24] and others.

2. Metric properties of the continued fraction expansions in (1.1)

Roughly speaking, the metrical theory of continued fraction expansions is about properties of the sequence $(a_n)_{n\in\mathbb{N}}$ and related sequences (see Section 3.2). The main purpose of this section is to determine the probability structure of $(a_n)_{n\in\mathbb{N}_+}$ under the Lebesgue measure λ. Before that, we shortly present the metrical theory of these continued fraction expansions. Another important result is the *Legendre-type theorem* (see, e.g., [3,11,15]) which is one of the main reasons for studying continued fractions, because it tells us that good approximations of irrational numbers by rational numbers are given by continued fraction convergents.

2.1. Some elementary properties of the continued fraction expansion in (1.1)

Here, we want to prove the convergence of expansion of the type of (1.1). First, note that in the rational case, the continued fraction expansion (1.1) is finite, unlike the irrational case, when we have an infinite number of non-negative digits.

Define $[a_1, a_2, \ldots, a_n]_m$ the convergent of $\omega \in \Omega$ by truncating the expansion on the right-hand side of (1.1). We want to show

$$\omega = \lim_{n\to\infty} [a_1, a_2, \ldots, a_n]_m, \quad \omega \in \Omega. \tag{2.1}$$

To this end, define integer-valued functions $p_n(\omega)$ and $q_n(\omega)$, for $n \in \mathbb{N}_+$, by

$$p_n(\omega) = m^{a_n} p_{n-1}(\omega) + (m-1)m^{a_{n-1}} p_{n-2}(\omega), \quad n \geqslant 2, \tag{2.2}$$

$$q_n(\omega) = m^{a_n} q_{n-1}(\omega) + (m-1)m^{a_{n-1}} q_{n-2}(\omega), \quad n \geqslant 1, \tag{2.3}$$

with $p_0(\omega) = 0$, $q_0(\omega) = 1$, $p_1(\omega) = 1$, $q_{-1}(\omega) = 0$ and $a_0 \equiv 0$.

Now, it is easy to prove by induction that for any $n \in \mathbb{N}_+$ we have

$$p_n(\omega)q_{n-1}(\omega) - p_{n-1}(\omega)q_n(\omega) = (-1)^{n-1}(m-1)^{n-1}m^{a_1+\cdots+a_{n-1}}, \tag{2.4}$$

and

$$\cfrac{m^{-a_1}}{1+\cfrac{(m-1)m^{-a_2}}{1+\ddots+\cfrac{(m-1)m^{-a_n}}{1+(m-1)t}}} = \frac{p_n(\omega)+(m-1)tm^{a_n}p_{n-1}(\omega)}{q_n(\omega)+(m-1)tm^{a_n}q_{n-1}(\omega)}, \tag{2.5}$$

with $0 \leqslant t \leqslant 1$.

It follows from the definitions of τ_m and a_n that for any $\omega \in \Omega$ we have

$$\tau_m^{n-1}(\omega) = \frac{m^{-a_n}}{1+(m-1)\tau_m^n(\omega)}, \quad n \in \mathbb{N}_+, \tag{2.6}$$

hence

$$\omega = \cfrac{m^{-a_1}}{1+\cfrac{(m-1)m^{-a_2}}{1+\ddots+\cfrac{(m-1)m^{-a_n}}{1+(m-1)\tau_m^n(\omega)}}}, \quad n\in\mathbb{N}_+. \tag{2.7}$$

By combining (2.7), (2.2) and (2.3) we have

$$\omega = \frac{p_n(\omega)+(m-1)\tau_m^n(\omega)m^{a_n}p_{n-1}(\omega)}{q_n(\omega)+(m-1)\tau_m^n(\omega)m^{a_n}q_{n-1}(\omega)}, \quad \omega\in\Omega,\ n\in\mathbb{N}_+. \tag{2.8}$$

Taking $\tau_m^n(\omega)=0$ in (2.8) gives

$$[a_1,a_2,\ldots,a_n]_m = \frac{p_n(\omega)}{q_n(\omega)}. \tag{2.9}$$

Now, using (2.4), (2.8) and (2.9), for any $\omega\in\Omega$ we obtain

$$\left|\omega-\frac{p_n(\omega)}{q_n(\omega)}\right| = \frac{(m-1)^n\tau_m^n(\omega)m^{a_1+\cdots+a_n}}{q_n(\omega)(q_n(\omega)+(m-1)\tau_m^n(\omega)m^{a_n}q_{n-1}(\omega))}, \quad n\in\mathbb{N}_+. \tag{2.10}$$

Note that this equation measures the difference between $\omega\in\Omega$ and its convergent and is the key ingredient of the following estimate.

Lemma 2.1. *For any $\omega\in\Omega$ we have*

$$\left|\omega-\frac{p_n(\omega)}{q_n(\omega)}\right| \leqslant \left(\frac{m-1}{m}\right)^n, \quad n\in\mathbb{N}_+. \tag{2.11}$$

Proof. By applying $\tau_m^n(\omega)\leqslant 1$ to (2.10), we have

$$\left|\omega-\frac{p_n(\omega)}{q_n(\omega)}\right| \leqslant \frac{(m-1)^n m^{a_1+\cdots+a_n}}{q_n(\omega)(q_n(\omega)+(m-1)m^{a_n}q_{n-1}(\omega))}, \quad n\in\mathbb{N}_+. \tag{2.12}$$

Let

$$t_n := \frac{(m-1)^n m^{a_1+\cdots+a_n}}{q_n(\omega)(q_n(\omega)+(m-1)m^{a_n}q_{n-1}(\omega))}, \quad n\in\mathbb{N}_+. \tag{2.13}$$

From (2.3), we have that $q_n(\omega)+(m-1)m^{a_n}q_{n-1}(\omega)\geqslant m\cdot m^{a_n}q_{n-1}(\omega)$, i.e., $q_n(\omega)\geqslant m^{a_n}q_{n-1}(\omega)$. Thus, by (2.13) and since $q_n(\omega)\geqslant q_{n-1}(\omega)+(m-1)m^{a_{n-1}}q_{n-2}(\omega)$, we have

$$\begin{aligned} t_n &\leqslant \frac{m-1}{m}\left(\frac{(m-1)^{n-1}m^{a_1+\cdots+a_{n-1}}}{q_n(\omega)q_{n-1}(\omega)}\right)\\ &\leqslant \frac{m-1}{m}\left(\frac{(m-1)^{n-1}m^{a_1+\cdots+a_{n-1}}}{q_{n-1}(\omega)(q_{n-1}(\omega)+(m-1)m^{a_{n-1}}q_{n-2}(\omega))}\right)\\ &= \frac{m-1}{m}t_{n-1}. \end{aligned} \tag{2.14}$$

Now, by direct computation, we have

$$t_1 \leqslant \frac{m-1}{m} m^{-a_1} \leqslant \frac{m-1}{m}$$

and (2.14) shows that $t_n \leqslant (\frac{m-1}{m})^n$, i.e., (2.11). □

Finally, (2.1) follows from (2.11), as $\frac{m-1}{m} < 1$.

2.2. Approximation result

Diophantine approximation (see, e.g., [15]) deals with the approximation of real numbers by rational numbers. Before we give the corresponding approximation result, we define the *cylinder* (or *fundamental interval*) of rank n, $I_m(i^{(n)})$, and show that any $I_m(i^{(n)})$ is the set of irrationals from a certain open interval with rational endpoints.

For any $n \in \mathbb{N}_+$ and $i^{(n)} = (i_1, \ldots, i_n) \in \mathbb{N}^n$ we will say that

$$I_m\big(i^{(n)}\big) = \big\{\omega \in \Omega\colon\ a_k(\omega) = i_k,\ 1 \leqslant k \leqslant n\big\} \tag{2.15}$$

is the *fundamental interval of rank n* and make the convention that $I_m(i^{(0)}) = \Omega$.

For example, for any $i \in \mathbb{N}$ we have

$$I_m(i) = \big\{\omega \in \Omega\colon\ a_1(\omega) = i\big\} = \Omega \cap \big(m^{-(i+1)}, m^{-i}\big). \tag{2.16}$$

We will write $I_m(a_1, \ldots, a_n) = I_m(a^{(n)})$, $n \in \mathbb{N}_+$. If $n \geqslant 2$ and $i_n \in \mathbb{N}$, then we have

$$I_m(a_1, \ldots, a_n) = I_m\big(i^{(n)}\big).$$

From the definition of τ_m and (2.8) we have

$$I_m\big(a^{(n)}\big) = \Omega \cap \big(u\big(a^{(n)}\big), v\big(a^{(n)}\big)\big), \tag{2.17}$$

where

$$u\big(a^{(n)}\big) = \begin{cases} \frac{p_n(\omega)+(m-1)m^{a_n}p_{n-1}(\omega)}{q_n(\omega)+(m-1)m^{a_n}q_{n-1}(\omega)}, & \text{if } n \text{ is odd,} \\ \frac{p_n(\omega)}{q_n(\omega)}, & \text{if } n \text{ is even,} \end{cases} \tag{2.18}$$

and

$$v\big(a^{(n)}\big) = \begin{cases} \frac{p_n(\omega)}{q_n(\omega)}, & \text{if } n \text{ is odd,} \\ \frac{p_n(\omega)+(m-1)m^{a_n}p_{n-1}(\omega)}{q_n(\omega)+(m-1)m^{a_n}q_{n-1}(\omega)}, & \text{if } n \text{ is even.} \end{cases} \tag{2.19}$$

Now, using (2.4), a direct computation shows that

$$\lambda\big(I\big(a^{(n)}\big)\big) = \frac{(m-1)^n m^{a_1+\cdots+a_n}}{q_n(\omega)(q_n(\omega)+(m-1)m^{a_n}q_{n-1}(\omega))} \tag{2.20}$$

and from (2.10) and (2.11) we have that

$$\lambda\big(I\big(a^{(n)}\big)\big) \leqslant \left(\frac{m-1}{m}\right)^n. \tag{2.21}$$

We now give a Legendre-type result for these continued fraction expansions. First we define the *approximation coefficient* $\Theta_m := \Theta_m(\omega)$ by

$$\Theta_m := q^2\left|\omega - \frac{p_n}{q_n}\right|, \quad n \in \mathbb{N}_+,$$

where $\frac{p_n}{q_n}$ is the nth continued fraction convergent of $\omega \in \Omega$. The approximation coefficient gives a numerical indication of the quality of the approximation.

Proposition 2.2. *For $\omega \in \Omega$ and p/q being a rational number with $p < q$, $q > 0$ and* g.c.d.$(p, q) = 1$, *let*

$$\frac{p}{q} = [i_1, \ldots, i_n]_m, \qquad \frac{p_{n-1}}{q_{n-1}} = [i_1, \ldots, i_{n-1}]_m$$

with $p_0 = 0$ and $q_0 = 1$, where the length $n = n(p/q) \in \mathbb{N}_+$ of the continued fraction expansion of p/q is chosen in such a way that it is even if $p/q < \omega$ and odd otherwise. Then

$$\Theta_m < \frac{(m-1)^n m^{i_1+\cdots+i_n} q}{q + (m-1)m^{i_n} q_{n-1}} \quad \textit{if and only if} \quad \frac{p}{q} \quad \textit{is a convergent of } \omega.$$

Proof. If p/q is a convergent of ω, then by (2.10) we have

$$\Theta_m = q^2\left|\omega - \frac{p}{q}\right| = \frac{(m-1)^n \tau_m^n(\omega) m^{i_1+\cdots+i_n} q}{q + (m-1)\tau_m^n(\omega) m^{i_n} q_{n-1}(\omega)} \leqslant \frac{(m-1)^n m^{i_1+\cdots+i_n} q}{q + (m-1)m^{i_n} q_{n-1}}.$$

Conversely, if $\Theta_m < \frac{(m-1)^n m^{i_1+\cdots+i_n} q}{q+(m-1)m^{i_n} q_{n-1}}$, then

$$q\left|\omega - \frac{p}{q}\right| < \frac{(m-1)^n m^{i_1+\cdots+i_n}}{q + (m-1)m^{i_n} q_{n-1}}.$$

Assuming that n is even, then $\omega > \frac{p}{q}$ and we have $\omega - \frac{p}{q} < \frac{(m-1)^n m^{i_1+\cdots+i_n}}{q(q+(m-1)m^{i_n} q_{n-1})}$. Thus,

$$\frac{p}{q} < \omega < \frac{p}{q} + \frac{(m-1)^n m^{i_1+\cdots+i_n}}{q(q + (m-1)m^{i_n} q_{n-1})} = \frac{p + (m-1)m^{i_n} p_{n-1}}{q + (m-1)m^{i_n} q_{n-1}}.$$

Hence, $\omega \in I_m(i^{(n)})$, i.e., $\frac{p}{q} = [i_1, \ldots, i_n]_m$ is a convergent of ω. The case when n is an odd is treated similarly. □

2.3. The probability structure of $(a_n)_{n\in\mathbb{N}_+}$ under the λ

We start by deriving the so-called Brodén–Borel–Lévy formula (see, e.g., [10,11]) for these type of expansions. First, define s_n, $n\in\mathbb{N}_+$, by

$$s_n = m^{-a_n}\frac{q_n}{q_{n-1}} - 1, \qquad s_1 = 0, \tag{2.22}$$

where $m\geqslant 2$ and a_n, q_n are defined in (1.3) and (2.3), respectively.

Next, (2.3) implies that

$$s_n = \frac{(m-1)m^{-a_n}}{1+s_{n-1}}, \quad n\geqslant 2, \tag{2.23}$$

hence

$$s_n = \cfrac{(m-1)m^{-a_n}}{1+\cfrac{(m-1)m^{-a_{n-1}}}{1+\ddots+\cfrac{(m-1)m^{-a_3}}{1+(m-1)m^{-a_2}}}} = (m-1)[a_n,a_{n-1},\ldots,a_2,\infty]_m, \tag{2.24}$$

for $n\geqslant 2$.

Proposition 2.3 *(Brodén–Borel–Lévy formula type). For any $n\in\mathbb{N}_+$ we have*

$$\lambda\big(\tau_m^n < x \,\big|\, a_1,\ldots,a_n\big) = \frac{(s_n+m)x}{(s_n+(m-1)x+1)}, \quad x\in I, \tag{2.25}$$

where s_n is defined by (2.22) or (2.23).

Proof. As we know, for any $n\in\mathbb{N}_+$ and $x\in I$, we have

$$\lambda\big(\tau_m^n < x \,\big|\, a_1,\ldots,a_n\big) = \frac{\lambda((\tau_m^n < x)\cap I_m(a_1,\ldots,a_n))}{\lambda(I_m(a_1,\ldots,a_n))}.$$

From (2.8) and (2.17) we have

$$\begin{aligned}\lambda\big(\big(\tau_m^n < x\big)\cap I(a_1,\ldots,a_n)\big) &= \left|\frac{p_n}{q_n} - \frac{p_n+(m-1)xm^{a_n}p_{n-1}}{q_n+(m-1)xm^{a_n}q_{n-1}}\right| \\ &= \frac{(m-1)^n x m^{a_1+\cdots+a_n}}{q_n(q_n+(m-1)xm^{a_n}q_{n-1})}.\end{aligned}$$

Hence, from (2.20) we have

$$\begin{aligned}\lambda\big(\tau_m^n < x \,\big|\, a_1,\ldots,a_n\big) &= \frac{\lambda((\tau_m^n < x)\cap I_m(a_1,\ldots,a_n))}{\lambda(I_m(a_1,\ldots,a_n))} \\ &= \frac{x(q_n+(m-1)m^{a_n}q_{n-1})}{q_n(q_n+(m-1)xm^{a_n}q_{n-1})} \\ &= \frac{(s_n+m)x}{s_n+(m-1)x+1},\end{aligned}$$

for any $n\in\mathbb{N}_+$ and $x\in I$. □

The Brodén–Borel–Lévy formula allows us to determine the probability structure of $(a_n)_{n\in\mathbb{N}_+}$ under λ.

Proposition 2.4. *For any $i \in \mathbb{N}$ and $n \in \mathbb{N}_+$ we have*

$$\lambda(a_1 = i) = (m-1)m^{-(i+1)} \tag{2.26}$$

and

$$\lambda(a_{n+1} = i \mid a_1, \ldots, a_n) = P_m^i(s_n), \tag{2.27}$$

where

$$P_m^i(x) = \frac{(m-1)m^{-(i+1)}(x+1)(x+m)}{(x+(m-1)m^{-i}+1)(x+(m-1)m^{-(i+1)}+1)}. \tag{2.28}$$

Proof. As shown above, we have

$$\left\{\omega \in \Omega\colon\ a_1(\omega) = i\right\} = \Omega \cap \left(m^{-(i+1)}, m^{-i}\right).$$

Thus,

$$\lambda(a_1 = i) = \left|m^{-(i+1)} - m^{-i}\right| = (m-1)m^{-(i+1)}.$$

From (2.6), we have that

$$\tau_m^n(\omega) = [a_{n+1}, a_{n+2}, \ldots]_m, \quad n \in \mathbb{N}_+,\ \omega \in \Omega,$$

and so we have

$$\begin{aligned}
\lambda(a_{n+1} = i \mid a_1, \ldots, a_n) &= \lambda\left(\tau_m^n \in \left(m^{-(i+1)}, m^{-i}\right] \,\middle|\, a_1, \ldots, a_n\right) \\
&= \frac{(s_n+m)m^{-i}}{s_n+(m-1)m^{-i}+1} - \frac{(s_n+m)m^{-(i+1)}}{s_n+(m-1)m^{-(i+1)}+1} \\
&= \frac{(m-1)m^{-(i+1)}(s_n+1)(s_n+m)}{(s_n+(m-1)m^{-i}+1)(s_n+(m-1)m^{-(i+1)}+1)} \\
&= P_m^i(s_n). \qquad \square
\end{aligned}$$

Hence, the sequence $(s_n)_{n\in\mathbb{N}_+}$ with $s_1 = 0$ is a homogeneous I-valued Markov chain on $(I, \mathcal{B}_I, \lambda)$ with the following transition mechanism: from state $s \in I \setminus \Omega$, $s \geqslant 1$ the only possible one-step transitions are those to states $m^{-i}/(1+(m-1)s)$, $i \in \mathbb{N}$, with corresponding probabilities $P_m^i(s)$, $i \in \mathbb{N}$.

2.4. The invariant measure of τ_m

In this subsection we will give the explicit form of the invariant probability measure γ_m of the transformation τ_m, i.e., $\gamma_m(A) = \gamma_m(\tau_m^{-1}(A))$, $A \in \mathcal{B}_I$.

Let $\mathcal{B}_I$ denote the σ-algebra of Borel subsets of I. The metric point of view in studying the sequence $(a_n)_{n\in\mathbb{N}_+}$ is to consider that the a_n, $n \in \mathbb{N}_+$, are non-negative integer-valued random variables which are defined almost surely on $(I, \mathcal{B}_I)$ with respect to any probability measure on $\mathcal{B}_I$ that assigns probability 0 to the set $I \setminus \Omega$ of rationals in I. Such a measure is a Lebesgue measure λ.

Another measure on $\mathcal{B}_I$ more important than Lebesgue measure, that assigns probability 0 to the set of rationals in I, is the *invariant probability measure* γ_m of the transformation τ_m.

Proposition 2.5. *The invariant probability density ρ_m of the transformation τ_m is given by*

$$\rho_m(x) = \frac{1}{((m-1)x+1)((m-1)x+m)}, \quad x \in I, \tag{2.29}$$

with the normalizing factor $k_m = \frac{(m-1)^2}{\log(m^2/(2m-1))}$.

Proof. See Appendix A. □

Hence

$$\gamma_m(A) = k_m \int_A \frac{dx}{((m-1)x+1)((m-1)x+m)}, \quad A \in \mathcal{B}_I. \tag{2.30}$$

The normalization constant k_m defined above is chosen so that $\gamma_m([0,1]) = 1$.

3. The natural extension of τ_m and extended random variables

By its very definition, the sequence $(a_n)_{n\in\mathbb{N}_+}$ in (1.3) and (1.4) is strictly stationary under γ_m. As such, there should exist a doubly infinite version of it, say $\bar{a}_l$, $l \in \mathbb{Z} := \{\ldots, -1, 0, 1, \ldots\}$, defined on a richer probability space. It appears that this doubly infinite version can be effectively constructed on $(I^2, \mathcal{B}_I^2, \overline{\gamma}_m)$, where $\overline{\gamma}_m$ is the so-called extended measure which expression is given below.

3.1. Definition and basic properties

For τ_m in (1.2), the *natural extension* $\overline{\tau}_m$ of τ_m [19] is the transformation of $[0,1) \times I$ defined by

$$\overline{\tau}_m(x,y) = \left(\tau_m(x), \frac{m^{-a_1(x)}}{(m-1)y+1}\right), \quad (x,y) \in [0,1) \times I. \tag{3.1}$$

This is a one-to-one transformation of Ω^2 with the inverse

$$\overline{\tau}_m^{-1}(\omega,\theta) = \left(\frac{m^{-a_1(\theta)}}{(m-1)\omega+1}, \tau_m(\theta)\right), \quad (\omega,\theta) \in \Omega^2. \tag{3.2}$$

It is easy to check that for $n \geqslant 2$ we have

$$\overline{\tau}_m^n(\omega,\theta) = \left(\tau_m^n(\omega), \left[a_n(\omega), a_{n-1}(\omega), \ldots, a_2(\omega), a_1(\omega) + \frac{\log(1+(m-1)\theta)}{\log m}\right]_m\right), \tag{3.3}$$

and

$$\overline{\tau}_m^{-n}(\omega,\theta)=\left(\left[a_n(\theta),a_{n-1}(\theta),\ldots,a_2(\theta),a_1(\theta)+\frac{\log(1+(m-1)\omega)}{\log m}\right]_m,\tau_m^n(\theta)\right). \tag{3.4}$$

Now, define the *extended measure* $\overline{\gamma}_m$ on $\mathcal{B}_I^2$ as

$$\overline{\gamma}_m(B)=k_m\iint_B\frac{dx\,dy}{((m-1)(x+y)+1)^2},\quad B\in\mathcal{B}_I^2. \tag{3.5}$$

A simple calculus shows us that

$$\overline{\gamma}_m(A\times I)=\overline{\gamma}_m(I\times A)=\gamma_m(A),\quad A\in\mathcal{B}_I. \tag{3.6}$$

The result below shows that $\overline{\gamma}_m$ plays with respect to $\overline{\tau}_m$ the part played by γ_m with respect to τ_m.

Proposition 3.1. *The extended measure $\overline{\gamma}_m$ is preserved by $\overline{\tau}_m$.*

Proof. See Appendix A. □

3.2. Extended random variables

Define extended incomplete quotients $\bar{a}_l$, $l\in\mathbb{Z}$, on Ω^2 by

$$\bar{a}_{l+1}(\omega,\theta)=\bar{a}_1\big(\overline{\tau}_m^l(\omega,\theta)\big),\quad l\in\mathbb{Z},$$

with

$$\bar{a}_1(\omega,\theta)=a_1(\omega),\quad (\omega,\theta)\in\Omega^2.$$

By (3.3) and (3.4) we have

$$\bar{a}_n(\omega,\theta)=a_n(\omega),\qquad \bar{a}_0(\omega,\theta)=a_1(\theta),\qquad \bar{a}_{-n}(\omega,\theta)=a_{n+1}(\theta),\quad n\in\mathbb{N}_+,\ (\omega,\theta)\in\Omega^2.$$

Remark 3.2. Since $\overline{\tau}_m$ preserves $\overline{\gamma}_m$, the doubly infinite sequence $(\bar{a}_l)_{l\in\mathbb{Z}}$ is strictly stationary under $\overline{\gamma}_m$.

Theorem 3.3. *For any $x\in I$ we have*

$$\overline{\gamma}_m\big([0,x]\times I\,\big|\,\bar{a}_0,\bar{a}_{-1},\ldots\big)=\frac{((m-1)a+m)x}{(m-1)(x+a)+1}\quad \overline{\gamma}_m\text{-a.s.}, \tag{3.7}$$

where $a=[\bar{a}_0,\bar{a}_{-1},\ldots]_m$.

Proof. Let $I_{m,n}$ denote the fundamental interval $I_m(\bar{a}_0,\bar{a}_{-1},\ldots,\bar{a}_{-n})$, $n\in\mathbb{N}$. We have

$$\overline{\gamma}_m\big([0,x]\times I\,\big|\,\bar{a}_0,\bar{a}_{-1},\ldots\big)=\lim_{n\to\infty}\overline{\gamma}_m\big([0,x]\times I\,\big|\,\bar{a}_0,\ldots,\bar{a}_{-n}\big)\quad \overline{\gamma}_m\text{-a.s.}$$

and

$$\begin{aligned}\overline{\gamma}_m\big([0,x]\times I \mid \bar{a}_0,\ldots,\bar{a}_{-n}\big) &= \frac{\overline{\gamma}_m([0,x]\times I_{m,n})}{\overline{\gamma}_m(I\times I_{m,n})}\\ &= \frac{k_m\int_{I_{m,n}} dy\int_0^x \frac{du}{((m-1)(u+y)+1)^2}}{\gamma_m(I_{m,n})}\\ &= \frac{1}{\gamma_m(I_{m,n})}k_m\int\limits_{I_{m,n}}\frac{x}{((m-1)(x+y)+1)((m-1)y+1)}\,dy\\ &= \frac{1}{\gamma_m(I_{m,n})}\int\limits_{I_{m,n}}\frac{x((m-1)y+m)}{(m-1)(x+y)+1}\gamma_m(dy)\\ &= \frac{x((m-1)y_n+m)}{(m-1)(x+y_n)+1},\end{aligned}$$

for some $y_n\in I_{m,n}$. Since

$$\lim_{n\to\infty} y_n = [\bar{a}_0,\bar{a}_{-1},\ldots]_m = a, \tag{3.8}$$

the proof is complete. □

The stochastic property of $(\bar{a}_l)_{l\in\mathbb{Z}}$ under $\overline{\gamma}_m$ is given by the following corollary of Theorem 3.3.

Corollary 3.4. *For any $i\in\mathbb{N}$ we have*

$$\overline{\gamma}_m(\bar{a}_1 = i \mid \bar{a}_0,\bar{a}_{-1},\ldots) = P_m^i\big((m-1)a\big)\quad \overline{\gamma}_m\text{-a.s.},$$

where $a=[\bar{a}_0,\bar{a}_{-1},\ldots]_m$.

Proof. Let us denote by $I_{m,n}$ the fundamental interval $I_m(\bar{a}_0,\bar{a}_{-1},\ldots,\bar{a}_{-n})$, $n\in\mathbb{N}$. We have

$$(\bar{a}_1 = i) = \big(m^{-(i+1)},m^{-i}\big]\times[0,1)$$

and

$$\overline{\gamma}_m(\bar{a}_1 = i \mid \bar{a}_0,\bar{a}_{-1},\ldots) = \lim_{n\to\infty}\overline{\gamma}_m(\bar{a}_1 = i \mid I_{m,n}).$$

Now

$$\begin{aligned}\overline{\gamma}_m\big(\big(m^{-(i+1)},m^{-i}\big)\times[0,1)\mid I_{m,n}\big) &= \frac{\overline{\gamma}_m((m^{-(i+1)},m^{-i})\times I_{m,n})}{\overline{\gamma}_m(I\times I_{m,n})}\\ &= \frac{1}{\gamma_m(I_{m,n})}\int\limits_{I_n} P_m^i\big((m-1)y\big)\gamma_m(dy)\\ &= P_m^i\big((m-1)y_n\big),\end{aligned}$$

for some $y_n\in I_{m,n}$. From (3.8) the proof is complete. □

Remark 3.5. The strict stationarity of $(\bar{a}_l)_{l\in\mathbb{Z}}$, under $\overline{\gamma}_m$ implies that

$$\overline{\gamma}_m(\bar{a}_{l+1} = i \mid \bar{a}_l, \bar{a}_{l-1}, \ldots) = P_m^i\big((m-1)a\big) \quad \overline{\gamma}_m\text{-a.s.}$$

for any $i \in \mathbb{N}$ and $l \in \mathbb{Z}$, where $a = [\bar{a}_l, \bar{a}_{l-1}, \ldots]_m$. The last equation emphasizes that $(\bar{a}_l)_{l\in\mathbb{Z}}$ is a chain of infinite order in the theory of dependence with complete connections (see [10, Section 5.5]).

Motivated by Theorem 3.3 we shall consider the family of (conditional) probability measures $(\gamma_m^a)_a$ on $\mathcal{B}_I$ defined by their distribution functions

$$\gamma_m^a\big([0,x]\big) = \frac{((m-1)a+m)x}{(m-1)(x+a)+1}, \quad x \in I,\ a \geqslant 0. \tag{3.9}$$

Note that the limit case $a = \infty$ is $\gamma_m^\infty = \lambda$.

For any $a \geqslant 0$ put $s_0^a = a$ and

$$s_n^a = \frac{(m-1)m^{-a_n}}{1+s_{n-1}^a}, \quad n \in \mathbb{N}_+. \tag{3.10}$$

For $a \geqslant 0$ we have

$$s_1^a = \frac{(m-1)m^{-a_1}}{1+a}$$

and

$$s_n^a = (m-1)\left[a_n, \ldots, a_2, a_1 + \frac{\log(a+1)}{\log m}\right]_m, \quad n \geqslant 2.$$

Then $(s_n^a)_{n\in\mathbf{N}_+}$ is an $I \cup \{a\}$-valued Markov chain on $(I, \mathcal{B}_I, \gamma_m^a)$ which starts from $s_0^a = a \geqslant 0$ and has the following transition mechanism: from state $s \in I \cup \{a\}$ the possible transitions are to any state $m^{-i}/((m-1)s+1)$ with the corresponding transition probability $P_m^i((m-1)s)$, $i \in \mathbb{N}$.

Now, it is easy to check by induction that

$$s_n^a = m^{-a_n}\frac{(m-1)p_n + (a+1)q_n}{(m-1)p_{n-1} + (a+1)q_{n-1}} - 1, \tag{3.11}$$

for any $n \in \mathbb{N}_+$ and $a \geqslant 0$.

Thus, a simple calculation shows that for any $n \in \mathbb{N}_+$ we have

$$\begin{aligned}\gamma_m^a\big(\tau_m^n < x \,\big|\, a_1, \ldots, a_n\big) &= \frac{\gamma_m^a((\tau_m^n < x) \cap I_m(a^{(n)}))}{\gamma_m^a(I_m(a^{(n)}))} \\ &= \frac{x((m-1)((m-1)p_n + (a+1)q_n) + m^{a_n}((m-1)p_{n-1} + (a+1)q_{n-1}))}{(m-1)((m-1)p_n + (a+1)q_n) + xm^{a_n}((m-1)p_{n-1} + (a+1)q_{n-1})}.\end{aligned}$$

By (3.11) for any $n \in \mathbb{N}_+$ we have

$$\gamma_m^a\big(\tau_m^n < x \,\big|\, a_1, \ldots, a_n\big) = \frac{((m-1)s_n^a + m)x}{(m-1)(x+s_n^a)+1}, \quad a \geqslant 0,\ x \in I. \tag{3.12}$$

The last equation is the generalization of the Brodén–Borel–Lévy formula from Section 2.3.

4. The Perron–Frobenius operator of τ_m under γ_m

In this section we derive and study the associated Perron–Frobenius operator of τ_m under the invariant measure γ_m.

Let μ be a probability measure on $(I, \mathcal{B}_I)$ such that $\mu(\tau_m^{-1}(A)) = 0$ whenever $\mu(A) = 0$, $A \in \mathcal{B}_I$, where the transformation τ_m is defined in (1.2). In particular, this condition is satisfied if τ_m is μ-preserving, that is, $\mu\tau_m^{-1} = \mu$. It is known from previous section, that the Perron–Frobenius operator P_μ of τ_m under μ is defined as the bounded linear operator on $L^1_\mu = \{f : I \to \mathbb{C} \mid \int_I |f| \, d\mu < \infty\}$ which takes $f \in L^1_\mu$ into $P_\mu f \in L^1_\mu$ with

$$\int_A P_\mu f \, d\mu = \int_{\tau_m^{-1}(A)} f \, d\mu, \quad A \in \mathcal{B}_I. \tag{4.1}$$

In particular, the Perron–Frobenius operator P_λ of τ_m under the Lebesgue measure λ is (see [4, p. 86])

$$P_\lambda f(x) = \frac{d}{dx} \int_{\tau_m^{-1}([0,x])} f \, d\lambda = \sum_{t \in \tau_m^{-1}(x)} \frac{f(t)}{|\tau_m'(t)|} \quad \text{a.e. in } I. \tag{4.2}$$

The following results will be proved in Appendix A.

The following proposition gives the expression of the Perron–Frobenius operator of τ_m under the invariant measure γ_m (4.3) and under a probability measure which is absolutely continuous with respect to the Lebesgue measure (4.6). Also, we derive the asymptotic behavior of this operator (4.8).

Proposition 4.1.

(i) *The Perron–Frobenius operator $U_m := P_{\gamma_m}$ of τ_m under γ_m is given a.e. in I by the equation*

$$U_m f(x) = \sum_{i \in \mathbb{N}} P_m^i\big((m-1)x\big) f\big(u_m^i(x)\big), \quad f \in L^1_{\gamma_m}, \tag{4.3}$$

where P_m^i is defined in (2.28) and $u_m^i(x)$ is given by the equation

$$u_m^i(x) = \frac{m^{-i}}{(m-1)x+1}, \quad x \in I. \tag{4.4}$$

(ii) *Let μ be a probability measure on $\mathcal{B}_I$. Assume that μ is absolutely continuous with respect to λ (and denote $\mu \ll \lambda$, i.e., if $\mu(A) = 0$ for every set A with $\lambda(A) = 0$) and let $h = d\mu/d\lambda$ a.e. in I. Then:*
(a) *The Perron–Frobenius operator P_μ of τ_m under μ is given a.e. in I by the equation*

$$P_\mu f(x) = \frac{1}{h(x)} \sum_{i \in \mathbb{N}} \frac{h(u_m^i(x))}{((m-1)x+1)^2} (m-1) m^{-i} f\big(u_m^i(x)\big) \tag{4.5}$$

$$= \frac{U_m g(x)}{((m-1)x+1)((m-1)x+m)h(x)}, \quad f \in L^1_\mu, \tag{4.6}$$

where $g(x) = ((m-1)x+1)((m-1)x+m) f(x)h(x)$, $x \in I$.
The powers of P_μ are given a.e. in I and for any $f \in L^1_\mu$ and any $n \in \mathbb{N}_+$ by the equation

$$P_\mu^n f(x) = \frac{U_m^n g(x)}{((m-1)x+1)((m-1)x+m)h(x)}. \tag{4.7}$$

(b) *We have*

$$\mu\left(\tau_m^{-n}(A)\right) = \int_A \frac{U_m^n f(x)}{((m-1)x+1)((m-1)x+m)}\,dx, \tag{4.8}$$

for any $n \in \mathbb{N}$ *and* $A \in \mathcal{B}_I$, *where* $f(x) = ((m-1)x+1)((m-1)x+m)h(x)$, $x \in I$.

In the next proposition the domain of U_m will be successively restricted to the following Banach spaces: $BV(I)$ is the linear space of all complex-valued functions of bounded variation and $B(I)$ is the collection of all bounded measurable functions $f : I \to \mathbb{C}$. The variation $\operatorname{var}_A f$ over $A \subset I$ of a function $f : I \to \mathbb{C}$ is defined as

$$\sup \sum_{i=1}^{k-1} \left| f(t_i) - f(t_{i-1}) \right|,$$

the supremum being taken over $t_1 < \cdots < t_k$, $t_i \in A$, $1 \leqslant i \leqslant k$, and $k \geqslant 2$. We write simply $\operatorname{var} f$ for $\operatorname{var}_I f$.

Proposition 4.2.

(i) *If* $f \in BV(I)$ *is a real-valued function, then*

$$\operatorname{var} U_m f \leqslant K_m \operatorname{var} f, \tag{4.9}$$

where $K_m = \frac{(m-1)(3m^2-3m+1)}{(2m-1)(m^2+m-1)}$. *The constant cannot be lowered.*

(ii) *The operator* $U_m : B(I) \to B(I)$ *is the transition operator of the Markov chain* $(s_n^a)_{n\in\mathbb{N}_+}$ *on* $(I, \mathcal{B}_I, \gamma_m^a)$, *for any* $a \in I$, *where* $(s_n^a)_{n\in\mathbb{N}_+}$ *and* γ_m^a *are give in* (3.9) *and* (3.10), *respectively.*

5. Proof of the Gauss–Kuzmin-type theorem

In this section we prove our main theorem. The main tool of this section is the random system with complete connections. We will first give a brief introduction to the theory of random systems with complete connections and list some of the main applications and some important properties. The general concepts presented here will be customized in the second subsection for the continued fraction expansion presented in this paper. All these concepts will be applied in Section 5.3 to solve our main theorem.

5.1. Random systems with complete connections

The purpose of this subsection is to recall the definition of random systems with complete connections, and take this opportunity to inform nonspecialists a little about some applications of the theory of random systems with complete connections.

The first explicit formal definition of the concept of dependence with complete connections was given by Onicescu and Mihoc in the 1930s when studying so-called *urn schemes* (see, e.g., [21], or [12] or the Introduction in [10]). The concept of random system with complete connections was defined by Iosifescu [9]. There are many other areas where the theory of RSCC can be applied. Let us just mention a few: *mathematical modeling of learning processes* (see, e.g., [20,12,14]), *chains of infinite order* (see, e.g., [6,7]), *partially observed random chains* (see, e.g., [12]), *image coding* (see [2]), and *continued fraction expansion* (see [10]). Nowadays RSCC are called *iterated functions systems with place-dependent probabilities* or simply *iterated functions systems* (IFS). This terminology was introduced by Barnsley

et al. in the middle of the 1980s in [1]. It only became fashionable in the framework of fractals and chaos but, before that, it appeared as the simplest case of a random system with complete connections and, in particular, as the Bush–Mosteller model for learning with experimenter-controlled-events [see, e.g., [2,8]]. An application of IFS to continued fractions can be found in the paper [18].

5.1.1. Definitions and explanations

First, let $(W, \mathcal{W})$ and $(X, \mathcal{X})$ be two measurable spaces. A real valued function P defined on $W \times \mathcal{X}$ is called a *transition probability function* from $(W, \mathcal{W})$ to $(X, \mathcal{X})$ if $P(w, \cdot)$ is a probability on $\mathcal{X}$ for any $w \in W$ and $P(\cdot, A)$ is a $\mathcal{W}$-measurable function for any $A \in \mathcal{X}$.

A quadruple

$$\big\{(W, \mathcal{W}), (X, \mathcal{X}), u, P\big\} \tag{5.1}$$

is named a *random system with complete connections* (RSCC) if

(i) $(W, \mathcal{W})$ and $(X, \mathcal{X})$ are measurable spaces;
(ii) $u : W \times X \to W$ is a $(\mathcal{W} \otimes \mathcal{X}, \mathcal{W})$-measurable function;
(iii) P is a transition probability function from $(W, \mathcal{W})$ to $(X, \mathcal{X})$.

The definition of an RSCC can be extended to the non-homogeneous case in the sense that all the entities constituting it are allowed to depend on $t \in T$, where T is either the set $\mathbb{N}$ of natural numbers or the set $\mathbb{Z}$ of integers.

The set W is usually called the *state space*, the set X is often called the *event space* and the function u is often called the *response-function*. We also call $u(\cdot, x) : W \to W$ a response-function.

The interpretation of this structure is as follows. If X denotes the set of possible observations and W the range of possible states of the system, then P induces for every state $w \in W$ the distribution $P(w, \cdot)$ of the random observation following w. The function u represents the transition function of the system, which transforms a given state w and an actual observation x into a new state $u(w, x)$.

To every RSCC $\{(W, \mathcal{W}), (X, \mathcal{X}), u, P\}$ and every $w \in W$ (an arbitrary fixed element of W) one can generate two stochastic sequences $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{\zeta_n\}_{n\in\mathbb{N}_+}$ as follows: we set $\xi_0 = w$, pick an element $\zeta_1 \in X$ using $P(\xi_0, \cdot)$, define $\xi_1 = u(\xi_0, \zeta_1)$, pick ζ_2 in X using $P(\xi_1, \cdot)$, define $\xi_2 = u(\xi_1, \zeta_2)$, and generally we pick ζ_n in X using $P(\xi_{n-1}, \cdot)$, and define $\xi_n = u(\xi_{n-1}, \zeta_n)$. Thus, the two stochastic sequences can be described as follows:

$$\xi_0 = w, \qquad \xi_{n+1} = u(\xi_n, \zeta_{n+1}), \quad n \geqslant 1,$$

$$P(\zeta_1 \in A) = P(w, A), \quad A \in \mathcal{X},$$

$$P(\zeta_{n+1} \in A \mid \xi_n, \zeta_n, \ldots, \xi_1, \zeta_1, \xi_0) = P(\xi_n, A), \quad A \in \mathcal{X}.$$

We call the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ of W-valued random variables the *state sequence* and the sequence $\{\zeta_n\}_{n\in\mathbb{N}_+}$ of X-valued random variables the *event sequence*. When we want to emphasize the initial point w, we write

$$\xi_n = \xi_n(w) \quad \text{and} \quad \zeta_n = \zeta_n(w).$$

The central issue in the theory of dependence with complete connections is the sequence $\{\zeta_n\}_{n\in\mathbb{N}_+}$ which is a stochastic process that is no longer Markovian, but a chain with complete connections (processes whose transition probabilities depend on the whole past history).

From the definition of ξ_n it is clear that the state sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is a Markov chain (the so-called associated Markov chain) with transition probability function Q, where

$$Q(w, A) = P\big(w, \big\{x \in X \,\big|\, u(w, x) \in A\big\}\big) \tag{5.2}$$

with $A \in \mathcal{W}$.

The transition operator $U : B(W, \mathcal{W}) \to B(W, \mathcal{W})$ is defined by

$$Uf(w) = \sum_{x \in X} P(w, x) f\big(u(w, x)\big), \quad f \in B(W, \mathcal{W}), \tag{5.3}$$

where $B(W, \mathcal{W})$ is the Banach space of all bounded $\mathcal{W}$-measurable complex-valued functions defined on W.

5.1.2. Examples of RSCCs

In this section we shall give two examples of RSCCs which occur either in various chapters of probability theory or as a result of modeling phenomena in various fields.

Example 5.1. The concept of a random system with complete connections may be regarded as a generalization and formalization of the notion of a stochastic learning model. Learning may be defined as an adaptive modification of behavior in the course of repeated trials. By mathematical learning theory we mean the body of research methods and results concerned with the conceptual representation of learning phenomena, the mathematical formulation of hypotheses about learning, and the derivation of testable theorems. The purpose of mathematical learning theory is to provide simple, quantitative descriptions of processes which are basic to behavioral modifications.

All stochastic models for learning studied so far fit the following general theoretical scheme. The behavior of the subject on trial n is determined by its state S_n (an indicator of the subject's tendencies) at the beginning of the trial. Here S_n is a random variable taking values in a measurable space $(S, \mathcal{S})$. On trial n an event E_{n+1} occurs that results in a change of the state. Here E_{n+1} is a random variable taking values in the measurable space $(E, \mathcal{E})$ and specifies those occurrences on trial n that affect the subsequent behavior. To represent the fact that the occurrence of an event affects a change of state it is necessary to consider a measurable map v from $S \times E$ into S and postulate that $S_{n+1} = v(S_n, E_{n+1})$, $n \in \mathbb{N}$. Finally assume that the probability distribution of E_{n+1} given $S_n, E_n, \ldots, S_1, E_1, S_0$ depends only on the state S_n and denote it by $R(S_n, \cdot)$. By a general learning model we mean the collection $\{(S, \mathcal{S}), (E, \mathcal{E}), v, R\}$ which is trivially an RSCC. Notice that in fact we only changed the notation. Various special learning models are obtained by simply particularizing S, E, v and R (see, e.g., [12,20]). □

Example 5.2. As we mentioned in Section 1.1, any irrational number y in the unit interval $[0, 1]$ has an infinite continued fraction expansion of the form

$$y = \cfrac{1}{a_1(y) + \cfrac{1}{a_2(y) + \cfrac{1}{a_3(y) + \ddots}}},$$

where the $a_n(y)$, $n \in \mathbb{N}_+$, are natural numbers. Define $(s_n)_{n \in \mathbb{N}_+}$ by

$$s_1 = \frac{1}{a_1}, \qquad s_{n+1} = \frac{1}{s_n + a_{n+1}}, \quad n \in \mathbb{N}_+.$$

Let us consider the RSCC $\{(W, \mathcal{W}), (X, \mathcal{X}), u, P\}$, where

$$W = [0, 1], \qquad \mathcal{W} = \mathcal{B}_{[0,1]},$$
$$X = \mathbb{N}_+, \qquad \mathcal{X} = \mathcal{P}_{\mathbb{N}_+},$$

$$u : W \times X \to W, \quad u(w,x) = \frac{1}{w+x},$$
$$P : W \times \mathcal{X} \to W, \quad P(w,x) = \frac{w+1}{(w+x)(w+x+1)}.$$

The sequences $(a_n)_{n\in\mathbb{N}_+}$ and $(s_n)_{n\in\mathbb{N}_+}$, $s_0 = 0$, are equivalent to the chain with complete connections $(\zeta_n)_{n\in\mathbb{N}_+}$ and the Markov chain $(\xi_n)_{n\in\mathbb{N}}$ associated with the above RSCC. More precisely, defining the one-to-one map θ from $(\mathbb{N}_+)^{\mathbb{N}_+}$ into $[0,1]$ by

$$\theta(a_1,a_2,a_3,\ldots) = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}, \quad a_i \in \mathbb{N}_+,\ i \in \mathbb{N}_+,$$

we have $\zeta_n(\sigma) = a_n(\theta(\sigma))$, $\xi_n(\sigma) = s_n(\theta(\sigma))$, $n \in \mathbb{N}_+$, $\sigma \in (\mathbb{N}_+)^{\mathbb{N}_+}$. □

5.1.3. Properties of the associated operators

In this subsection we present the asymptotic and ergodic properties of the associated operators. These properties are used to obtain the ergodicity of an RSCC by letting the associated Markov chain satisfy some topological properties. To state these results we need some preliminary definitions.

Let Q_n be the transition probability function defined by

$$Q_n(w,A) = \frac{1}{n}\sum_{k=1}^{n} Q^k(w,A)$$

where Q^k, $k \geqslant 1$, is the k-step transition probability function of the Markov chain associated with RSCC (5.1). Let U_n be the Markov operator associated with Q_n.

Next, let us consider the norm $\|\cdot\|_L$ defined on $L(W) =$ the space of Lipschitz complex-valued functions defined on W by

$$\|f\|_L = \sup_{w\in W}\left|f(w)\right| + \sup_{w'\neq w''}\frac{|f(w') - f(w'')|}{|w' - w''|}, \quad f \in L(W).$$

As is well known, $(L(W), \|\cdot\|_L)$ is a Banach space.

The following can be found in [10].

If there exists a linear bounded operator U^∞ from $L(W)$ to $L(W)$ such that

$$\lim_{n\to\infty}\left\|U_n f - U^\infty f\right\|_L = 0,$$

for any $f \in L(W)$ with $\|f\|_L = 1$, we say U is *ordered*.

If

$$\lim_{n\to\infty}\left\|U^n f - U^\infty f\right\|_L = 0,$$

for any $f \in L(W)$ with $\|f\|_L = 1$, we say U is *aperiodic*, where U^n is the nth iterate of U, $n \in \mathbb{N}$, with U^0 is the identity.

If U is ordered and $U^\infty(L(W))$ is one-dimensional space, it is named *ergodic* with respect to $L(W)$.

If U is ergodic and aperiodic, it is named *regular* with respect to $L(W)$ and the corresponding Markov chain has the same name.

The definition below is due to M.F. Norman [20] and isolates a class of RSCCs, called *RSCCs with contraction*.

An RSCC $\{(W,\mathcal{W}),(X,\mathcal{X}),u,P\}$ is said to be an RSCC *with contraction* if and only if there is a distance d on W and the metric space (W,d) is separable, $r_1<\infty$, $R_1<\infty$, and there exists a natural integer k such that $r_k<1$, where

$$r_k=\sup_{w'\neq w''}\sum_{X^k}P_k\big(w,x^{(k)}\big)\frac{\mathrm{d}(w'x^{(k)},w''x^{(k)})}{\mathrm{d}(w',w'')},\quad k\in\mathbb{N}_+,$$

and

$$R_k=\sup_{A\in\mathcal{X}^k}\sup_{w'\neq w''}\frac{P_k(w',A)-P_k(w'',A)}{\mathrm{d}(w',w'')}.$$

The following result can be found in [10].

Theorem 5.3. *Let W be a compact metric space with a distance* d *and $\{(W,\mathcal{W}),(X,\mathcal{X}),u,P\}$ be an RSCC with contraction.*

(i) *The Markov chain associated to the RSCC is regular if and only if there exists a point $w_0\in W$ such that*

$$\lim_{n\to\infty}\mathrm{d}\big(\sigma_n(w),w_0\big)=0,$$

for any $w\in W$, where $\sigma_n(w)=\operatorname{supp}Q^n(w,\cdot)$ ($\operatorname{supp}\mu$ denotes the support of the measure μ).

(ii) *The supports of $Q^n(w,\cdot)$, $n\in\mathbb{N}_+$, $w\in W$, can be iteratively computed as follows*:

$$\sigma_{m+n}(w)=\overline{\bigcup_{w'\in\sigma_m(w)}\sigma_n\big(w'\big)},$$

for any $m,n\in\mathbb{N}_+$, $w\in W$, where the overline means the topological closure.

An RSCC $\{(W,\mathcal{W}),(X,\mathcal{X}),u,P\}$, whose associated Markov chain is regular with respect to $B((W,\mathcal{W}))$, is *uniformly ergodic* and $\lim_{n\to\infty}\varepsilon_n=0$, where

$$\varepsilon_n:=\sup_{\substack{w\in W,r\in\mathbb{N}_+\\A\in\mathcal{X}^r}}\big|P_r^n(w,A)-\mathrm{P}_r^\infty(A)\big|,$$

while P_r^∞ is the probability on $\mathcal{X}^r$.

Theorem 5.4. *Let W be a compact metric space with a distance* d. *If the RSCC $\{(W,\mathcal{W}),(X,\mathcal{X}),u,P\}$ with contraction has regular associated Markov chain, then it is uniformly ergodic.*

5.2. The RSCC associated with expansion of the type of (1.1)

First, it is easy to check that P_m^i from (2.28) defines a transition probability function from $(I,\mathcal{B}_I)$ to $(\mathbb{N},\mathcal{P}(\mathbb{N}))$, i.e., $\sum_{i\in\mathbb{N}}P_m^i(x)=1$, $x\in I$.

Let us consider the random system with complete connections

$$\big\{(I,\mathcal{B}_I),\big(\mathbb{N}_+,\mathcal{P}(\mathbb{N}_+)\big),u,P\big\},\tag{5.4}$$

where $u:I\times\mathbb{N}\to I$, $u(x,i)=u_m^i(x)$ is given in (4.4) and the function $P(x,i)=P_m^i(x)$ given in (2.28).

We denote by U_m the associated Markov operator of RSCC (5.4) with the transition probability function

$$Q_m(x, A) = \sum_{\{i\in\mathbb{N}:\ u_m^i(x)\in A\}} P_m^i(x), \quad x \in I,\ A \in \mathcal{B}_I.$$

Then $Q_m^n(\cdot,\cdot)$ will denote the n-step transition probability function of the same Markov chain.

The ergodic behavior of RSCC (5.4) allows us to find the limiting distribution function F and the invariant measure Q_m^∞ induced by F.

Proposition 5.5. *RSCC (5.4) is uniformly ergodic.*

Proof. We apply Theorem 5.4. Putting $\Delta_i = m^{-i} - m^{-2i}$, $i \in \mathbb{N}$, we get

$$P_m^i(x) = (m-1)\left[m^{-(i+1)} + \frac{\Delta_i}{x + (m-1)m^{-i} + 1} - \frac{\Delta_{i+1}}{x + (m-1)m^{-(i+1)} + 1}\right].$$

We have

$$\frac{d}{dx}u(x,i) = -\frac{(m-1)m^{-i}}{((m-1)x+1)^2},$$
$$\frac{d}{dx}P(x,i) = (m-1)\left[\frac{\Delta_{i+1}}{(x+(m-1)m^{-(i+1)}+1)^2} - \frac{\Delta_i}{(x+(m-1)m^{-i}+1)^2}\right],$$

for all $x \in I$ and $i \in \mathbb{N}$, so that $\sup_{x\in I}|\frac{d}{dx}u(x,i)| = (m-1)m^{-i}$ and $\sup_{x\in I}|\frac{d}{dx}P(x,i)| < \infty$. Hence the requirements of definition of an RSCC with contraction are fulfilled. To prove the regularity of U with respect to $L(I)$ let us define recursively $x_{n+1} = (x_n+2)^{-1}$, $n \in \mathbb{N}$, with $x_0 = x$.

A criterion of regularity is expressed in Theorem 5.3(i), in terms of supports $\sigma_n(x)$ of the n-step transition probability functions $Q_m^n(x,\cdot)$, $n \in \mathbb{N}_+$. Clearly $x_{n+1} \in \sigma_1(x_n)$ and therefore Theorem 5.3(ii) and an induction argument lead us to the conclusion that $x_n \in \sigma_n(x)$, $n \in \mathbb{N}_+$. But, $\lim_{n\to\infty} x_n = \sqrt{2}-1$ for any $x \in I$. Hence

$$\mathrm{d}\big(\sigma_n(x), \sqrt{2}-1\big) \leqslant |x_n - \sqrt{2} + 1| \to 0 \quad \text{as } n \to \infty,$$

where $\mathrm{d}(x,y) = |x-y|$, for any $x, y \in I$. The regularity of U_m with respect to $L(I)$ follows from Theorem 5.3. Moreover, $Q_m^n(\cdot,\cdot)$ converges uniformly to a probability measure Q_m^∞ and there exist two positive constants $q < 1$ and k such that

$$\big\|U_m^n f - U_m^\infty f\big\|_L \leqslant kq^n \|f\|_L, \quad n \in \mathbb{N}_+,\ f \in L(I), \tag{5.5}$$

where

$$U_m^n f(\cdot) = \int_I f(y)\, Q_m^n(\cdot, dy), \tag{5.6}$$

$$U_m^\infty f = \int_I f(y)\, \mathrm{Q}_m^\infty(dy), \tag{5.7}$$

and Q_m^∞ is the invariant probability measure of the transformation τ_m, i.e., Q_m^∞ has the density $\rho_m(x)$ given in (2.29), $x \in I$. □

Now we are able to find the limiting distribution function

$$F(x) = F_\infty(x) = \lim_{n\to\infty} \mu\left(\tau_m^n < x\right)$$

and obtain a convergence rate result.

5.3. Proof of Theorem 1.1

We prove Theorem 1.1 in this subsection.

Proof of Theorem 1.1. By (5.7) we have

$$U_m^\infty f_0 = \int_I f_0(y) \mathrm{Q}_m^\infty(dx) = k_m, \quad f_0 \in L(I).$$

Taking into account (5.5), there exist two constants $q < 1$ and k such that

$$\left\| U_m^n f_0 - U_m^\infty f_0 \right\|_L \leqslant k q^n \| f_0 \|_L, \quad n \in \mathbb{N}_+.$$

Further, consider $C(I)$ the metric space of real continuous functions defined on I with the supremum norm $\|f\| = \sup_{x\in I} |f(x)|$. Since $L(I)$ is a dense subset of $C(I)$ we have

$$\lim_{n\to\infty} \left\| \left(U_m^n - U_m^\infty\right) f_0 \right\| = 0, \tag{5.8}$$

for all $f_0 \in C(I)$. Therefore, (5.8) is valid for a measurable function f_0 which is Q_m^∞-almost surely continuous, that is, for a Riemann-integrable function f_0. Thus, we have

$$\begin{aligned} F(x) = \lim_{n\to\infty} \mu\left(\tau_m^n < x\right) &= \lim_{n\to\infty} \int_0^x U_m^n f_0(u) \rho_m(u)\, du \\ &= k_m \int_0^x \rho_m(u)\, du \\ &= \frac{k_m}{(m-1)^2} \log \frac{m((m-1)x+1)}{(m-1)x+m}. \end{aligned}$$

Hence (1.6) is proved. □

Acknowledgments

The author would like to express their sincere thanks to the referees for their valuable comments.

Appendix A. Proofs of propositions

We prove Propositions 2.5, 3.1, 4.1 and 4.2 in this appendix.

Proof of Proposition 2.5. We briefly give some general properties about the Perron–Frobenius operator (see, e.g., [4,11]) which will be useful both to demonstrate this proposition and in Section 4.

Let $(X, \mathcal{X}, \mu)$ be a probability space. A transformation τ of X is said to be *μ-non-singular* if and only if $\mu(\tau^{-1}(A)) = 0$ for all $A \in \mathcal{X}$ for which $\mu(A) = 0$; it is said to be *measure-preserving* if and only if $\mu\tau^{-1} = \mu$, i.e., $\mu\tau^{-1}(A) = \mu(A)$ for all $A \in \mathcal{X}$. Clearly, any μ-preserving transformation is μ-non-singular.

The Perron–Frobenius operator P_μ associated with a μ-non-singular transformation τ is defined as the linear bounded operator on $L^1_\mu = \{f : I \to \mathbb{C}\colon \int_I |f|\,d\mu < \infty\}$ which takes $f \in L^1_\mu$ into $P_\mu f \in L^1_\mu$ with

$$\int_A P_\mu f\,d\mu = \int_{\tau^{-1}(A)} f\,d\mu, \quad A \in \mathcal{X},$$

or, equivalently

$$\int_X g P_\mu f\,d\mu = \int_X (g \circ \tau) f\,d\mu$$

for all $f \in L^1_\mu$ and $g \in L^\infty_\mu$.

In particular, the Perron–Frobenius operator P_λ of τ under the Lebesgue measure λ is (see [4, p. 86])

$$P_\lambda f(x) = \frac{d}{dx} \int_{\tau^{-1}([0,x])} f\,d\lambda = \sum_{t \in \tau^{-1}(x)} \frac{f(t)}{|\tau'(t)|} \quad \text{a.e. in } I. \tag{A.1}$$

The probabilistic interpretation of P_μ is immediate: if an X-valued random variable ξ on X has μ-density h, that is, $\mu(\xi \in A) = \int_A h\,d\mu$, $A \in X$, with $h \geqslant 0$ and $\int_X h\,d\mu = 1$, then $\tau \circ \xi$ has μ-density $P_\mu h$. The following properties hold:

(i) P_μ is positive, that is, $P_\mu f \geqslant 0$ if $f \geqslant 0$;
(ii) P_μ preserves integrals, that is, $\int_X P_\mu f\,d\mu = \int_X f\,d\mu$, $f \in L^1_\mu$;
(iii) $\|P_\mu\|_{p,\mu} := \sup(\|P_\mu f\|_{p,\mu}\colon\ f \in L^p_\mu,\ \|f\|_{p,\mu} = 1) \leqslant 1$ for any $p \geqslant 1$ and $p = \infty$;
(iv) for any $n \in \mathbb{N}_+$ the nth power P^n_μ of P_μ is the Perron–Frobenius operator associated with the nth iterate τ^n of τ under μ;
(v) $(P_\mu f)^* = P_\mu f^*$ for any $f \in L^1_\mu$, where z^* = complex conjugate of $z \in \mathbb{C}$ (= the set of complex numbers);
(vi) $P_\mu((g \circ \tau) f) = g P_\mu f$ for any $f \in L^1_\mu$ and $g \in L^\infty_\mu$;
(vii) $P_\mu f = f$ if and only if τ is ν-preserving, where ν is defined by $\nu(A) = \int_A f\,d\mu$, $A \in \mathcal{X}$. In particular, $P_\mu 1 = 1$ if and only if τ is μ-preserving.

Proof of Proposition 2.5 From above, it is sufficient to show that the function ρ_m defined in (2.29) is an eigenfunction of the Perron–Frobenius operator of τ_m with the eigenvalue 1:

$$P_{\tau_m} \rho_m(x) = \sum_{t \in \tau_m^{-1}(x)} \frac{\rho_m(t)}{|\tau_m'(t)|}. \tag{A.2}$$

First, we note that

$$\tau_m^{-1}(x) = \left\{ \frac{m^{-i}}{1 + (m-1)x}\colon\ i \geqslant 1,\ x \in I \right\}. \tag{A.3}$$

Thus

$$\begin{aligned}
P_{\tau_m}\rho_m(x) &= \sum_{i=0}^{\infty} \frac{(m-1)m^{-i}}{(1+(m-1)x)^2}\,\rho_m\left(\frac{m^{-i}}{1+(m-1)x}\right)\\
&= \sum_{i=0}^{\infty}(m-1)m^{-(i+1)}\,\frac{1}{((m-1)x+(m-1)m^{-(i+1)}+1)}\,\frac{1}{((m-1)x+(m-1)m^{-i}+1)}\\
&= \frac{1}{m-1}\sum_{i=0}^{\infty}\left(\frac{1}{(m-1)x+(m-1)m^{-(i+1)}+1}-\frac{1}{(m-1)x+(m-1)m^{-i}+1}\right)\\
&= \frac{1}{m-1}\left(\frac{1}{(m-1)x+1}-\frac{1}{(m-1)x+m}\right)\\
&= \frac{1}{((m-1)x+1)((m-1)x+m)} = \rho_m(x). \qquad \square
\end{aligned}$$

Proof of Proposition 3.1. We should show that $\overline{\gamma}_m(\overline{\tau}_m^{-1}(B)) = \overline{\gamma}_m(B)$ for any $B \in \mathcal{B}_I^2$ or, equivalently, since $\overline{\tau}_m$ is invertible on Ω^2, that

$$\overline{\gamma}_m\big(\overline{\tau}_m(B)\big) = \overline{\gamma}_m(B), \quad \text{for any } B \in \mathcal{B}_I^2. \tag{A.4}$$

We start with $B = (a,b)\times(c,d)$, where

$$a = m^{-(i+1)}, \qquad b = m^{-i}, \quad i \in \mathbb{N},$$

and c and d are the arbitrary numbers from $(0,1)$. Then

$$\overline{\tau}_m(B) = \left\{\left(\tau_m(x), \frac{m^{-a_1(x)}}{(m-1)y+1}\right) \,\middle|\, x \in (a,b),\ y \in (c,d)\right\}. \tag{A.5}$$

Taking $x = m^{-(i+\theta)}$, $0<\theta<1$, we have

$$\tau_m(x) = \frac{m^{\theta}-1}{m-1}, \qquad a_1(x) = i$$

such that

$$\overline{\tau}_m(B) = \left((0,1), \left(\frac{m^{-i}}{(m-1)d+1}, \frac{m^{-i}}{(m-1)c+1}\right)\right). \tag{A.6}$$

Let

$$I(m,i,c,d) \equiv \left(\frac{m^{-i}}{(m-1)d+1}, \frac{m^{-i}}{(m-1)c+1}\right).$$

A simple computation yields

$$\overline{\gamma}_m\big(\overline{\tau}_m(B)\big) = k_m \int_0^1 dx \int_{I(m,i,c,d)} \frac{dy}{((m-1)(x+y)+1)^2}$$
$$= k_m \int_{m^{-(i+1)}}^{m^{-i}} dx \int_c^d \frac{dy}{((m-1)(x+y)+1)^2} = \overline{\gamma}_m(B)$$

that is, (A.4) holds.

Next, we consider the case

$$a = \frac{m^{-i}}{(m-1)m^{-j}+1}, \qquad b = \frac{m^{-i}}{(m-1)m^{-(j+1)}+1}, \quad i, j \in \mathbb{N},$$

and (c, d) is an arbitrary interval. Now, with

$$x = \frac{m^{-i}}{(m-1)m^{-(j+\theta)}+1},$$

we have

$$\left\{\frac{\log x^{-1}}{\log m}\right\} = \left\{i + \frac{\log(1+(m-1)m^{-(j+\theta)})}{\log m}\right\} = \frac{\log(1+(m-1)m^{-(j+\theta)})}{\log m}$$

and

$$a_1(x) = \left\lfloor \frac{\log x^{-1}}{\log m} \right\rfloor = i.$$

Thus,

$$(m-1)\tau_m(x) = m^{\frac{\log(1+(m-1)m^{-(j+\theta)})}{\log m}} - 1 = (m-1)m^{-(j+\theta)}.$$

Hence,

$$\overline{\tau}_m(B) = \big(m^{-(j+1)}, m^{-j}\big) \times \left(\frac{m^{-i}}{(m-1)d+1}, \frac{m^{-i}}{(m-1)c+1}\right). \tag{A.7}$$

A straightforward calculation shows us that

$$\overline{\gamma}_m\big(\overline{\tau}_m(B)\big) = k_m \int_{m^{-(j+1)}}^{m^{-j}} dx \int_{I(m,i,c,d)} \frac{dy}{((m-1)(x+y)+1)^2}$$
$$= k_m \int_{I(m,i,m^{-j},m^{-(j+1)})} dx \int_c^d \frac{dy}{((m-1)(x+y)+1)^2} = \overline{\gamma}_m(B)$$

that is, (A.4) holds.

Since any arbitrary interval (a,b) can be written as a reunion of fundamental intervals the proof is complete. □

Proof of Proposition 4.1. (i) Let $\tau_{m,i}: I_i \to I$ denote the restriction of τ_m to the interval $I_i = (m^{-(i+1)}, m^{-i}]$, $i \in \mathbb{N}$, that is,

$$\tau_{m,i}(x) = \frac{1}{m-1}\left(\frac{m^{-i}}{x} - 1\right), \quad x \in I_i. \tag{A.8}$$

For any $f \in L^1_{\gamma_m}$ and any $A \in \mathcal{B}_I$, we have

$$\int\limits_{\tau_m^{-1}(A)} f\,d\gamma_m = \sum_{i\in\mathbb{N}} \int\limits_{\tau_m^{-1}(A\cap I_i)} f\,d\gamma_m = \sum_{i\in\mathbb{N}} \int\limits_{\tau_{m,i}^{-1}(A)} f\,d\gamma_m. \tag{A.9}$$

For any $i \in \mathbb{N}$, by the change of variable

$$x = \tau_{m,i}^{-1}(y) = \frac{m^{-i}}{(m-1)y+1}, \tag{A.10}$$

we successively obtain

$$\begin{aligned}
\int\limits_{\tau_{m,i}^{-1}(A)} f\,d\gamma_m &= k_m \int\limits_{\tau_{m,i}^{-1}(A)} \frac{f(x)}{((m-1)x+1)((m-1)x+m)}\,dx \\
&= k_m \int\limits_A \frac{f(u_m^i(y))}{((m-1)u_m^i(y)+1)((m-1)u_m^i(y)+m)} \frac{(m-1)m^{-i}}{((m-1)y+1)^2}\,dy \\
&= k_m \int\limits_A f\big(u_m^i(y)\big)(m-1)m^{-(i+1)} \frac{1}{((m-1)y+(m-1)m^{-i}+1)} \\
&\quad \times \frac{1}{((m-1)y+(m-1)m^{-(i+1)}+1)}\,dy \\
&= \int\limits_A P_m^i\big((m-1)y\big) f\big(u_m^i(y)\big)\gamma_m(dy).
\end{aligned} \tag{A.11}$$

Now, (4.3) follows from (A.9) and (A.11). □

(ii)(a) From (A.8) and (A.10), for any $f \in L^1_{\gamma_m}$ and any $A \in \mathcal{B}_I$, we have

$$\begin{aligned}
\int\limits_{\tau_m^{-1}(A)} f\,d\mu &= \sum_{i\in\mathbb{N}} \int\limits_{\tau_m^{-1}(A\cap I_i)} f\,d\mu = \sum_{i\in\mathbb{N}} \int\limits_{\tau_{m,i}^{-1}(A)} f\,d\mu \\
&= \sum_{i\in\mathbf{N}} \int\limits_{\tau_{m,i}^{-1}(A)} f(x)h(x)\,dx = \sum_{i\in\mathbf{N}} \int\limits_A \frac{f(u_m^i(y))h(u_m^i(y))(m-1)m^{-i}}{((m-1)y+1)^2}\,dy \\
&= \int\limits_A \sum_{i\in\mathbf{N}} \frac{h(u_m^i(x))}{((m-1)x+1)^2}(m-1)m^{-i} f\big(u_m^i(x)\big)\,dx.
\end{aligned} \tag{A.12}$$

Since $d\mu = h\,d\lambda$, (4.5) follows from (A.12). Now, since $g(x) = ((m-1)x+1)((m-1)x+m)f(x)h(x)$, from (4.3) we have

$$U_m g(x) = \frac{((m-1)x+m)}{(m-1)x+1}(m-1)\sum_{i\in\mathbb{N}} m^{-i} h\big(u_m^i(x)\big) f\big(u_m^i(x)\big). \tag{A.13}$$

Now, (4.6) follows immediately from (4.5) and (A.13). □

(b) We will use mathematical induction. For $n = 0$, Eq. (4.8) reduces to

$$\mu(A) = \int_A h(x)\,dx, \quad A \in \mathcal{B}_I,$$

which is obviously true. Assume that (4.8) holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned}\mu\big(\tau_m^{-(n+1)}(A)\big) &= \mu\big(\tau_m^{-n}\big(\tau_m^{-1}(A)\big)\big)\\ &= \int_{\tau_m^{-1}(A)} \frac{U_m^n f(x)}{((m-1)x+1)((m-1)x+m)}\,dx\\ &= \frac{1}{k_m}\int_{\tau_m^{-1}(A)} U_m^n f(x)\,d\gamma_m(x).\end{aligned}$$

By the very definition of the Perron–Frobenius operator $U_m = P_{\gamma_m}$ we have

$$\int_{\tau_m^{-1}(A)} U_m^n f\,d\gamma_m = \int_A U_m^{n+1} f\,d\gamma_m.$$

Therefore,

$$\begin{aligned}\mu\big(\tau_m^{-(n+1)}(A)\big) &= \frac{1}{k_m}\int_A U_m^{n+1} f\,d\gamma_m\\ &= \int_A \frac{U_m^{n+1} f(x)}{((m-1)x+1)((m-1)x+m)}\,dx\end{aligned}$$

which ends the proof. □

Proof of Proposition 4.2. (i) For $x, y \in I$ we have

$$\begin{aligned}U_m f(x) - U_m f(y) &= \sum_{i\in\mathbb{N}}\big(P_m^i\big((m-1)x\big) f\big(u_m^i(x)\big) - P_m^i\big((m-1)y\big) f\big(u_m^i(y)\big)\big)\\ &= \sum_{i\in\mathbb{N}}\big(P_m^i\big((m-1)x\big) - P_m^i\big((m-1)y\big)\big)\big(f\big(u_m^i(x)\big) - f\big(u_m^0(x)\big)\big)\\ &\quad + \sum_{i\in\mathbb{N}} P_m^i\big((m-1)y\big)\big(f\big(u_m^i(x)\big) - f\big(u_m^i(y)\big)\big)\end{aligned}$$

$$= \sum_{i\in\mathbb{N}_+} \big(P_m^i\big((m-1)x\big) - P_m^i\big((m-1)y\big)\big)\big(f\big(u_m^i(x)\big) - f\big(u_m^0(x)\big)\big)$$
$$+ \sum_{i\in\mathbb{N}} P_m^i\big((m-1)y\big)\big(f\big(u_m^i(x)\big) - f\big(u_m^i(y)\big)\big).$$

Note that the function P_m^0 is increasing, while the functions P_m^i, $i\in\mathbb{N}_+$, are all decreasing. Let $x<y$, with $x, y\in I$. It follows from the above equation that

$$\begin{aligned}\big|U_m f(x) - U_m f(y)\big| &\leqslant \left(\sum_{i\in\mathbb{N}_+} \big(P_m^i\big((m-1)x\big) - P_m^i\big((m-1)y\big)\big)\right)\operatorname{var} f\\ &\quad + \sup_{y\in I, i\in\mathbb{N}} P_m^i\big((m-1)y\big)\sum_{i\in\mathbb{N}} \operatorname{var}_{[x,y]} f\circ u_i(x)\\ &= \big(1 - P_m^0\big((m-1)x\big) - 1 + P_m^0\big((m-1)y\big)\big)\operatorname{var} f\\ &\quad + P_m^0(m-1)\sum_{i\in\mathbb{N}} \operatorname{var}_{[x,y]} f\circ u_i(x).\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{var} U_m f &\leqslant \big(2P_m^0(m-1) - P_m^0(0)\big)\operatorname{var} f = \left(\frac{2m(m-1)}{m^2+m-1} - \frac{m-1}{2m-1}\right)\operatorname{var} f\\ &= \frac{(m-1)(3m^2-3m+1)}{(2m-1)(m^2+m-1)}\operatorname{var} f.\end{aligned}$$

Define f by $f(x)=0$, $0\leqslant x\leqslant \frac{1}{m}$, and $f(x)=1$, $\frac{1}{m}<x\leqslant 1$. Then we have $U_m f(x) = P_m^0(x)$, $0\leqslant x<1$ and $U_m f(1)=0$. Since $\operatorname{var} U_m f = \frac{(m-1)(3m^2-3m+1)}{(2m-1)(m^2+m-1)}$ and $\operatorname{var} f = 1$, it follows that the constant K_m cannot be lowered. □

(ii) The transition operator of $(s_n^a)_{n\in\mathbb{N}_+}$ takes $f\in B(I)$ to the function defined by

$$\begin{aligned}E_a\big(f\big(s_{n+1}^a\big)\,\big|\, s_n^a = s\big) &= \sum_{i\in\mathbb{N}} P_m^i\big((m-1)s\big) f\big(u_m^i(s)\big)\\ &= U_m f(s),\quad s\in I,\end{aligned} \tag{A.14}$$

where E_a stands for the mean value operator with respect to the probability measure γ_m^a. □

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J. Math. Anal. Appl. 444 (2016) 610–623

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Dependence with complete connections and the Gauss–Kuzmin theorem for N-continued fractions

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ARTICLE INFO

Article history:
Received 18 November 2015
Available online 1 July 2016
Submitted by U. Stadtmueller

Keywords:
Continued fractions
Invariant measure
Random system with complete connections

ABSTRACT

We consider a family $\{T_N : N \geq 1\}$ of interval maps as generalizations of the Gauss transformation. For the continued fraction expansion arising from T_N, we solve its Gauss–Kuzmin-type problem by applying the theory of random systems with complete connections by Iosifescu.

1. Introduction

The purpose of this paper is to prove a Gauss–Kuzmin-type problem for N-continued fraction expansions introduced by Burger et al. [3]. In order to solve the problem, we apply the theory of random systems with complete connections by Iosifescu [9].

Fix an integer $N \geq 1$. In this paper, we consider a generalization of the Gauss transformation, i.e.,

$$T_N(x) := \begin{cases} \dfrac{N}{x} - \left\lfloor \dfrac{N}{x} \right\rfloor, & x \in I := [0,1], x \neq 0, \\ 0, & x = 0 \end{cases} \tag{1.1}$$

where $\lfloor \cdot \rfloor$ denotes the floor (or entire) function.

The generalized Gauss measure $G_N(A) = \dfrac{1}{\log\{(N+1)/N\}} \displaystyle\int_A \frac{dx}{x+N}$, $A \in \mathcal{B}_I = \sigma$-algebra of all Borel subsets of $[0,1]$ is T_N-invariant, i.e., $G_N\left(T_N^{-1}(A)\right) = G_N(A)$ for any $A \in \mathcal{B}_I$. Define $a_1(x) = \lfloor N/x \rfloor$,

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http://dx.doi.org/10.1016/j.jmaa.2016.06.046
0022-247X/

$x \in (0,1]$, $a_1(0) = \infty$, and $a_n(x) = a_1\left(T_N^{n-1}(x)\right)$, $x \in I$, $n \in \mathbb{N}_+ := \{1, 2, \ldots\}$, with $T_N^0(x) = x$. By the very definitions, Burger et al. proved in [3] that any irrational $0 < x < 1$ can be written in the form

$$x = \cfrac{N}{a_1 + \cfrac{N}{a_2 + \cfrac{N}{a_3 + \ddots}}} := [a_1, a_2, a_3, \ldots]_N \tag{1.2}$$

where a_n's are non-negative integers. We will call (1.2) the *N-continued fraction expansion of x*. In [4], Dajani et al. proved that $(I, \mathcal{B}_I, G_N, T_N)$ is an ergodic dynamical system.

The Perron–Frobenius operator of T_N under a non-atomic probability measure μ on $\mathcal{B}_{\mathcal{I}}$ (i.e., $\mu(T_N^{-1}(A)) = 0$ whenever $\mu(A) = 0$) is defined as the bounded linear operator U on the Banach space $L^1(I, \mu) := \{f : I \to \mathbb{C} : \int_I |f| d\mu < \infty\}$ such that the following holds:

$$\int_A Uf \, d\mu = \int_{T_N^{-1}(A)} f \, d\mu \quad \text{for all } A \in \mathcal{B}_I,\ f \in L^1(I, \mu). \tag{1.3}$$

In particular, the Perron–Frobenius operator of T_N under G_N is

$$\{Uf\}(x) = \sum_{i \geq N} V_{N,i}(x) \, f\left(\frac{N}{x+i}\right), \quad f \in L^1(I, G_N) \tag{1.4}$$

where $V_{N,i} := \dfrac{x+N}{(x+i)(x+i+1)}$ for any $i \geq N$ and $n \in \mathbb{N}_+$ [17]. An important fact is that for any probability measure μ on $\mathcal{B}_I$ such that $\mu \ll \lambda$, where λ is the Lebesgue measure on $\mathcal{B}_I$, we have

$$\mu\left(T_N^{-n}(A)\right) = \int_A \{U^n f\}(x) dG_N(x) \tag{1.5}$$

where $f(x) := \left(\log\left(\frac{N+1}{N}\right)\right)(x+N)h(x)$ for $x \in I$ [17].

The problem of finding the asymptotic of $T_N^{-n}(A)$ as $n \to \infty$ represents the Gauss–Kuzmin-type problem for N-continued fraction expansions.

Theorem 1.1 *(A Gauss–Kuzmin theorem for T_N). Fix an integer $N \geq 1$ and let $(I, \mathcal{B}_I, T_N)$ be as above.*

(i) *For a probability measure μ on $(I, \mathcal{B}_I)$, let the assumption (A) as follows:*

(A) *μ is non-atomic and has a Riemann-integrable density.*

Then for any probability measure μ which satisfies (A), the following holds:

$$\lim_{n\to\infty} \mu(G_N^n < x) = \frac{1}{\log\{(N+1)/N\}} \log\frac{x+N}{N}, \quad x \in I. \tag{1.6}$$

(ii) *In addition to assumptions of μ in (i), if the density of $I \ni x \mapsto \mu([0,x])$ is Lipschitz continuous, then there exist two positive constants $q < 1$ and ℓ such that for any $x \in I$ and $n \geq 1$, there exists θ with $|\theta| \leq \ell$, the following holds:*

$$\mu\left(G_N^n < x\right) = \frac{1 + \theta q^n}{\log\{(N+1)/N\}} \log\frac{x+N}{N} \tag{1.7}$$

where $\theta := \theta(N, \mu, n, x)$. *As a consequence, the* n*-th error term* $e_n(N, \mu; x)$ *of the Gauss–Kuzmin problem is obtained as follows:*

$$e_n(N, \mu; x) = \frac{\theta q^n}{\log\{(N+1)/N\}} \log \frac{x+N}{N}. \tag{1.8}$$

The paper is organized as follows. In Section 2, we recall random system with complete connections (= RSCC), and show examples and properties. In Section 3, we prove Theorem 1.1. By using the ergodic behavior of the RSCC associated with N-continued fraction expansion, we determine the limit of the sequence $(\,\mu(G_N^n < x)\,)_{n\geq 1}$ of distributions as $n \to \infty$.

2. Random systems with complete connections

In this section, we introduce random systems with complete connections and show its properties.

The first explicit formal definition of the concept of dependence with complete connections was given by Onicescu and Mihoc [22]. It is a non-trivial extension of Markovian dependence theory, and it was also investigated by Doeblin and Fortet [5] and by Harris [7]. The concept of random system with complete connections (= RSCC) was defined by Iosifescu [8].

Examples of RSCC are urns models [22,10], stochastic learning processes [21,10,12], partially observed random chains [10], image coding [1], continued fraction expansions [9,16,23–25] and others.

An RSCC is often called an *iterated function system with place-dependent probabilities* or simply an *iterated function system (= IFS)* [2]. Applications of IFS to continued fractions can be found in [15,19]. For more detail, see [9–11].

2.1. Definitions and their extensions

In this subsection, we introduce the definition of random system with complete connections which is restricted to a smaller class than the original. Next we extend domains of functions in such a system.

Definition 2.1. [8,9,21] A random system with complete connections (= RSCC) is a quadruple

$$\{(W, \mathcal{W}), X, u, P\} \tag{2.1}$$

where

(i) $(W, \mathcal{W})$ is a measurable space and X is a non-empty set;
(ii) $u : W \times X \to W$ is a measurable function with respect to W for each element in X;
(iii) $P : W \times X \to [0, 1]$ satisfies that $\sum_{x\in X} P(w, x) = 1$ for each $w \in W$, and $P(\cdot, x)$ is a measurable function on $(W, \mathcal{W})$ for each $x \in X$.

For an RSCC in Definition 2.1, we call W, X, u and P, the *state space*, the *event space*, the *transition function* and the *((W, X)-) transition probability function*, respectively ([21], Section 1.2). The role of the function u is to change a state $w \in W$ into the new state $w' = u(w, x) \in W$ by an event $x \in X$:

$$W \ni w \overset{x}{\mapsto} w' = u(w, x) \in W. \tag{2.2}$$

In this case, $P(w, x)$ is regarded as the probability of the transition $w \mapsto w'$ which depends on the information of both w and x.

Remark 2.2.

(i) In the original definition of RSCC in [9], X is assumed as a measurable space $(X, \mathcal{X})$. In Definition 2.1, we treat only the case $\mathcal{X} = \mathcal{P}(X)$ (= the power set of X). A lot of examples in [9] satisfy this condition.

(ii) An RSCC can be regarded as an automaton with output [20,13]. A *Mealy machine* $(W, X, \Delta, u, \lambda)$ consists of three sets W, X, Δ and two maps u, λ where W is the *set of states*, X is the *input alphabet*, Δ is the *output alphabet*, and the *transition function* $u : W \times X^* \to W$ and the *map of outputs* $\lambda : W \times X^* \to \Delta^*$ which satisfy

$$\begin{cases} u(w, xa) = u(u(w, x), a), \\ \lambda(w, xa) = \lambda(w, x)\lambda(u(w, x), a), \end{cases} \quad x \in X^*,\ a \in X, w \in W \tag{2.3}$$

where X^* and Δ^* denote free semigroups generated by X and Δ, respectively. When it is an RSCC, $\Delta = [0, 1]$ and Δ^* is reduced to the $[0, 1]$ with respect to the multiplication in $(\mathbb{R}, \times)$, $\pi : \Delta^* \to [0, 1]$. The map $\pi \circ \lambda$ is a transition probability function P. An example of Mealy machine as an RSCC will be shown in Example 2.5. The similarity can be explained as the reason that initial applications of RSCC were related to learning processes which may be understood as an algorithmic study of such systems.

For a given RSCC $\{(W, \mathcal{W}), X, u, P\}$, we extend domains of both u and P as follows.

(i) In order to extend the domain of u, we prepare the free semigroup X^* generated by X as follows: In Definition 2.1, we regard X as a set of letters, and treat X^n as the set of all strings of length $n \geq 1$. We write an element $(x_1, \ldots, x_n) \in X^n$ as $x_1 \cdots x_n$. Then the set X^* of all strings with the empty letter ε is a semigroup with the concatenation as its product, and the unit of X^* is ε. The semigroup X^* is called the *free semigroup generated by* X [18].
With respect to the semigroup X^*, the function u induces a unique right action of X^* on W which is denoted by the same symbol u here:

$$u : W \times X^* \to W; \quad (w, x) \mapsto wx := u(w, x). \tag{2.4}$$

For example, $w(xx')$ is defined as $(wx)x' = u(u(w, x), x')$ for $w \in W$, $x \in X^*$ and $x' \in X$. For $x = x_1 \cdots x_n \in X^n$, we can write $wx := w(x_1 \cdots x_n)$ for $w \in W$.

(ii) Let $\mathcal{P}(X)$ denote the power set of X. The domain $W \times X$ of P is extended to $W \times \mathcal{P}(X)$ as follows:

$$P(w, A) := \begin{cases} \sum_{x \in A} P(w, x), & A \neq \emptyset, \\ 0, & A = \emptyset, \end{cases} \quad (w, A) \in W \times \mathcal{P}(X). \tag{2.5}$$

By Definition 2.1(iii), we see that $(X, \mathcal{P}(X), P(w, \cdot))$ is a probability space for each $w \in W$.

As a generalization of P, we define $P_r : W \times X^r \to [0, 1]$ for each $r \geq 1$ by

$$P_1 := P, \quad P_r(w, xx') := P_{r-1}(w, x)P(wx, x'), \tag{2.6}$$

with $w \in W$, $x \in X^{r-1}$, $x \in X$, $r \geq 2$, where the notation in (2.4) is used. Then we see that $P_r(w, x)$ means the transition probability from w to $w \cdot x$ along the path $w \to wx_1 \to wx_1x_2 \to \cdots \to wx_1 \cdots x_r$ with length r. For $A \subset X^r$, we also define

$$P_r(w, A) := \begin{cases} \sum_{x \in A} P_r(w, x), & A \neq \emptyset, \\ 0, & A = \emptyset. \end{cases} \tag{2.7}$$

By definition, $P_r(w, A)$ is the summation of all paths from w to $w \cdot x$ for $x \in A$. For example, we can verify that $P_r(w, A \times X) = P_{r-1}(w, A)$ for any $(w, A) \in W \times \mathcal{P}(X^{r-1})$ by Definition 2.1(iii) and (2.6). Especially, $P_r(w, X^r) = 1$. Hence $(X^r, \mathcal{P}(X^r), P_r(w, \cdot))$ is also a probability space for each $w \in W$ and $r \geq 1$.

2.2. Transition probability functions defined on the square of the state space

Let $\{(W, \mathcal{W}), X, u, P\}$ be as in Definition 2.1. Recall that P is the transition probability function with the domain $W \times \mathcal{P}(X)$. We define new transition probability functions with the domain $W \times \mathcal{W}$ in this subsection.

2.2.1. Space of paths

For $w, w' \in W$, define the subset $X(w, w')$ of X by

$$X(w, w') := \{x \in X : u(w, x) = w'\}. \tag{2.8}$$

Then $X(w, w')$ can be identified with the set of all paths from w to w' with length 1. Remark that $X(w, w')$ may be the empty set. By definition, $X(w, w') \cap X(w, w'') = \emptyset$ when $w' \neq w''$ and $\coprod_{w' \in W} X(w, w') = X$. From these properties, the following holds.

Fact 2.3. *For $(w, B) \in W \times \mathcal{W}$, let $X(w, B) := \{x \in X : u(w, x) \in B\}$. Then the following holds for each $w \in W$:*

(i) $X(w, B) \cap X(w, B') = \emptyset$ *when* $B \cap B' = \emptyset$.
(ii) $X(w, B) = \bigcup_{w' \in B} X(w, w')$.
(iii) $X(w, B) \subset X(w, B')$ *when* $B \subset B'$.
(iv) $X(w, B) \cup X(w, B') = X(w, B \cup B')$.

In [9], the symbol $X(w, B)$ is written as B_w.

2.2.2. Transition probability functions defined on the square of the state space

By using the extension of P in (2.5) and $X(w, w')$ in (2.8), define the new function $Q : W \times W \to [0, 1]$ by

$$Q(w, w') := P(w, X(w, w')), \quad (w, w') \in W \times W. \tag{2.9}$$

This means the total sum of transition probabilities from w to w' by all possible event $x \in X$. We extend the domain of Q to $W \times \mathcal{W}$ as follows ([9], (1.1.11)):

$$Q(w, B) := P(w, X(w, B)) = \sum_{x \in X(w, B)} P(w, x), \quad (w, B) \in W \times \mathcal{W}. \tag{2.10}$$

We call Q the *((W, W)-) transition probability function* associated with $\{(W, \mathcal{W}), X, u, P\}$. The number $Q(w, B)$ means the probability such that a state w changes to a state belonging to B by measuring all possible (one-step) events. For $w \in W$, define the function Q_w as

$$Q_w : \mathcal{W} \to [0, 1]; \quad Q_w(B) := Q(w, B), \quad B \in \mathcal{W}. \tag{2.11}$$

By Fact 2.3,

$$0 \leq Q_w(B) \leq Q_w(W) = 1,$$

$$Q_w(B) + Q_w(B') = Q_w(B \cup B') \text{ when } B \cap B' = \emptyset. \tag{2.12}$$

Hence $(W, \mathcal{W}, Q_w)$ is a probability space for each $w \in W$. Remark that Q_w can be defined on the whole of $\mathcal{P}(W)$, but we restrict Q_w on the given σ-algebra $\mathcal{W} \subset \mathcal{P}(W)$ here. Thanks to Q_w, we can define the integration $\int_B f(w')\, dQ_w(w')$ for $B \in \mathcal{W}$ and a measurable function f on $(W, \mathcal{W})$.

For $w, w' \in W$, define the family $\{Q^{(k)}(w, w') : k \geq 1\}$ recursively as

$$Q^{(1)}(w, w') := Q(w, w'), \quad Q^{(k)}(w, w') := \int_W dQ_w(w'')\, Q^{(k-1)}(w'', w'), \quad k \geq 2. \tag{2.13}$$

By definition, $Q^{(k)}(w, w')$ is the expectation value of $Q^{(k-1)}(\cdot, w')$ on the probability space $(W, \mathcal{W}, Q_w)$. We extend the domain of $Q^{(k)}$ to $W \times \mathcal{W}$ as follows:

$$Q^{(1)}(w, B) := Q(w, B), \quad Q^{(k)}(w, B) := \int_W dQ_w(w')\, Q^{(k-1)}(w', B), \quad k \geq 2. \tag{2.14}$$

We will say $Q^{(k)}$ the *k-step transition probability function of the Markov chain* associated with $\{(W, \mathcal{W}), X, u, P\}$. We see that $\left(W, \mathcal{W}, Q_w^{(k)}\right)$ is also a probability space for each $w \in W$ and $k \geq 1$ where $Q_w^{(k)} := Q^{(k)}(w, \cdot)$. Furthermore, define

$$Q_n(w, B) := \frac{1}{n} \sum_{k=1}^{n} Q^{(k)}(w, B), \quad (w, B) \in W \times \mathcal{W},\ n \geq 1. \tag{2.15}$$

For example, $Q_1 = Q$. Then $(W, \mathcal{W}, Q_{n,w})$ is also a probability space for each $w \in W$ where $Q_{n,w}(B) := Q_n(w, B)$ for $B \in \mathcal{W}$.

2.3. Examples

In this section, we shall give two examples of RSCC.

Example 2.4. We show the RSCC associated with regular continued fraction expansions. More precisely, it is the RSCC associated with the dynamical system $(I, \mathcal{B}_I, \tau)$ where τ is the Gauss transformation, defined as $\tau = T_1$, where T_N is as in (1.1). Define the RSCC $\{(W, \mathcal{W}), X, u, P\}$ as $W = [0, 1]$, $\mathcal{W} = \mathcal{B}_{[0,1]}$, $X = \mathbb{N}_+$,

$$u : W \times X \to W; \quad u(w, x) = \frac{1}{w + x}, \tag{2.16}$$

$$P : W \times X \to [0, 1]; \quad P(w, x) = \frac{w + 1}{(w + x)(w + x + 1)}. \tag{2.17}$$

By definition, we see that $u(\cdot, x)$ is also a right inverse of τ for each $x \in X$, that is, $\tau(u(w, x)) = w$ for any $w \in W$. This shows that the dynamical system $(I, \mathcal{B}_I, \tau)$ is encoded onto the RSCC $\{(W, \mathcal{W}), X, u, P\}$. About more details, see Sec. 1.2 of [9].

Example 2.5. According to §3.1 of [13], we show a simple (but non-trivial) example of (finite) RSCC by using a finite automaton with input/output. Define the data $\{(W, \mathcal{W}), X, u, P\}$ as follows:

(i) For the 2-point set $\{1, 2\}$, let $\mathcal{P}(\{1, 2\})$ denote its power set. Then $(\{1, 2\}, \mathcal{P}(\{1, 2\}))$ is a measurable space. Define $(W, \mathcal{W}) := (\{1, 2\}, \mathcal{P}(\{1, 2\}))$ and $X := \{1, 2\}$.

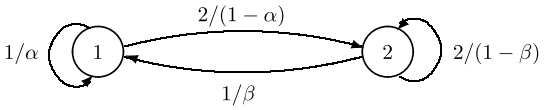

Fig. 1. RSCC as Mealy machine.

(ii) Any function $u : W \times X \to W$ is measurable with respect to $(W, \mathcal{W})$ by definition. For example, let

$$u(i,j) := j, \quad i,j = 1,2. \tag{2.18}$$

(iii) Any transition probability function $P : W \times X \to [0,1]$ is uniquely defined by two real numbers $P(i,1) \in [0,1]$ for $i = 1,2$. For example, for $0 \leq \alpha, \beta \leq 1$, define $P(1,1) = \alpha$ and $P(2,1) = \beta$. Then they define a unique transition probability function P as

$$P(1,1) = \alpha, \quad P(1,2) = 1-\alpha, \quad P(2,1) = \beta, \quad P(2,2) = 1-\beta. \tag{2.19}$$

For example, the value $Q(1,1)$ of Q in (2.9) is computed as follows:

$$Q(1,1) = P(1, \{x \in \{1,2\} : u(1,x) = 1\}) = P(1,1) = \alpha. \tag{2.20}$$

Remark that u in (2.18) is the case such that $u(i,\cdot)$ does not depend on $i \in W$ ([9], p. 15, (i)). As the case such that $u(i,\cdot)$ depends on $i \in W$, we can define other u, for example, $u(i,j) = i$ for $i \in W$, $j \in X$.

We can illustrate this example as a finite automaton with input/output (= Mealy machine [20]). Assume that $\{(W, \mathcal{W}), X, u, P\}$ is as in (2.18) and (2.19). For this system, we draw the transition diagram (= an oriented graph with labeling edges) as follows:

(1) The set of vertices is the state space $W = \{1,2\}$.
(2) For two vertices $i, j \in W$ (i and j may be same), if there exists an event (= input signal) $k \in X = \{1,2\}$ such that $u(i,k) = j$, then draw the oriented edge from i to j. We write (i,k,j) as this edge here.
(3) Write "$k/P(i,k)$" as the label of an edge (i,k,j).

According to these rules, the transition diagram of the RSCC is illustrated as Fig. 1.

For example, from (2.18), $u(1,1) = 1$, and from (2.19), $P(1,1) = \alpha$. Hence the label of the edge $(1,1,1)$ is given as "$1/\alpha$". About a Markov chain related to this example, see Chap. 5 of [6]. About other examples of Mealy machine, see [13,14].

2.4. Operators associated with an RSCC

In this subsection, we present the asymptotic and ergodic properties of operators associated with an RSCC. To state these results, we prepare definitions. We add the following assumption for all RSCC in this subsection:

Assumption (B). For an RSCC $\{(W, \mathcal{W}), X, u, P\}$, W is a measurable subset of the measurable space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ and $\mathcal{W} = \mathcal{B}_W$.

Thanks to Assumption (B), we can use the absolute value $|\cdot|$ and the Lebesgue measure on W induced by $\mathbb{R}$, which will be necessary in this subsection.

Let $L^\infty(W)$ denote the Banach space of all complex-valued bounded measurable functions defined on $(W, \mathcal{W})$. We define operators on $L^\infty(W)$ associated with an RSCC $\{(W, \mathcal{W}), X, u, P\}$ as follows.

Definition 2.6.

(i) The *transition operator* U on $L^\infty(W)$ is defined by

$$\{Uf\}(w) := \sum_{x\in X} P(w,x)\, f(u(w,x)), \quad f \in L^\infty(W),\ w \in W. \tag{2.21}$$

(ii) ([9], (3.1.7)) For $n \geq 1$, define the operator U_n on $L^\infty(W)$ as

$$\{U_n f\}(w) := \int_W f(w')\, dQ_{n,w}(w'), \quad f \in L^\infty(W),\ w \in W \tag{2.22}$$

where $Q_{n,w} := Q_n(w,\cdot)$ is as in (2.15).

Remark 2.7.

(i) Let U be as in (2.21). For each $w \in W$, $\{U(f)\}(w)$ is the expectation value of $(f\circ u)(w,\cdot)$ with respect to the probability space $(X, \mathcal{P}(X), P(w,\cdot))$. For example, if f is the characteristic function of $B \in \mathcal{W}$, then we see $\{Uf\}(w) = Q(w,B)$. For each $n \geq 1$, $\{U^n f\}(w) = \sum_{x\in X^n} P_n(w,x) f(wx)$ where U^n denotes the n-the iterate of U and P_n is as in (2.6) for $r = n$.

(ii) Let U_n be as in (2.22). Remark that U_n is well-defined on $L^\infty(W)$ because $Q_{n,w}$ is a probability measure. For example, if f is the characteristic function of $B \in \mathcal{W}$, then $\{U_1 f\}(w) = Q(w,B)$. Since

$$Q_{n,w}(w') = Q_n(w,w') = \frac{1}{n}\sum_{k=1}^{n} Q^{(k)}(w,w'), \tag{2.23}$$

we see that

$$\{U_n f\}(w) = \frac{1}{n}\sum_{k=1}^{n}\int_W f(w')\, dQ_w^{(k)}(w'). \tag{2.24}$$

Next, let $L(W)$ denote the Banach space of all complex-valued Lipschitz continuous functions on W with the following norm $\|\cdot\|_L$:

$$\|f\|_L := \sup_{w\in W}|f(w)| + \sup_{w'\neq w''}\frac{|f(w') - f(w'')|}{|w' - w''|}, \quad f \in L(W). \tag{2.25}$$

Remark that we use Assumption (B) here. By definition, $L(W) \subset L^\infty(W)$.

According to [9,21], we introduce several characterizations of the operator U in (2.21) as follows.

Definition 2.8. ([9], Definition 3.1.4, [21], Definition 2.1) Let $W, U, U_n, L(W)$ be as in (2.1), (2.21), (2.22) and (2.25), respectively. We consider restrictions of U, U_n on $L(W)$ as follows.

(i) U is ordered if there exists a bounded linear operator S on $L(W)$ such that

$$\lim_{n\to\infty}\|U_n f - Sf\|_L = 0, \quad f \in L(W). \tag{2.26}$$

(ii) U is aperiodic if there exists a bounded linear operator S' on $L(W)$ such that

$$\lim_{n\to\infty}\|U^n f - S' f\|_L = 0, \quad f \in L(W), \tag{2.27}$$

where U^n is the n-th iterate of U for $n \geq 1$.

(iii) U is ergodic with respect to $L(W)$ if U is ordered and the rank of S in (2.26) is 1.
(iv) U is regular with respect to $L(W)$ if U is ergodic and aperiodic.
(v) The Markov chain corresponding to U is regular if U is regular with respect to $L(W)$.

Remark 2.9.

(i) Definition 2.8(i) and (ii) mean that sequences (U_n) and (U^n) converge to operators S and S', respectively, with respect to the strong operator topology on $L(W)$.
(ii) Under the Assumption (B), W is separable. In addition, if W is complete and U is orderly, then there exists probability measures $\{Q_w^\infty : w \in W\}$ on $(W, \mathcal{W})$ such that

$$\{Sf\}(w) = \int_W dQ_w^\infty(w')\, f(w'), \quad f \in L(W),\, w \in W \tag{2.28}$$

by Theorem 3.1.24 of [9] where S is as in Definition 2.8(i).
(iii) In addition to the assumption in (ii), if U is ergodic with respect to $L(W)$, then Sf in (2.28) is a constant function on W for any $f \in L(W)$. Therefore the operator S is identified with a bounded linear functional on $L(W)$, $S : L(W) \to \mathbb{C}$. Then (2.28) is rewritten as

$$S : L(W) \to \mathbb{C}; \quad Sf = \int_W dQ^\infty(w')\, f(w'), \quad f \in L(W) \tag{2.29}$$

for some probability measure Q^∞ on $(W, \mathcal{W})$.

Definition 2.10. ([9], Definition 3.1.15) Under the Assumption (B), $\{(W, \mathcal{W}), X, u, P\}$ is an RSCC with contraction if the following conditions are satisfied:

(i) $r_1 < \infty$,
(ii) $r_\ell < 1$ for some $\ell \geq 1$, and
(iii) $R < \infty$

where (r_k) and R are defined as

$$r_k := \sup_{w' \neq w''} \sum_{x \in X^k} P(w', x) \frac{|w'x - w''x|}{|w' - w''|}, \quad k \geq 1, \tag{2.30}$$

$$R := \sup_{A \subset X} \sup_{w' \neq w''} \frac{|P(w', A) - P(w'', A)|}{|w' - w''|}. \tag{2.31}$$

Remark that we use the Assumption (B) for $|\cdot|$ in Definition 2.10.
When $k = 1$, (2.30) is rewritten by using u as follows:

$$r_1 = \sup_{w' \neq w''} \sum_{x \in X} P(w', x) \frac{|u(w', x) - u(w'', x)|}{|w' - w''|}. \tag{2.32}$$

If $\sup_{w' \neq w''} |u(w', x) - u(w'', x)|/|w' - w''| < 1$, then $r_1 < 1$. In this case, assumptions (i) and (ii) in Definition 2.10 are satisfied.

Theorem 2.11. *Under the Assumption (B), let $\{(W,\mathcal{W}),X,u,P\}$ be an RSCC with contraction. For $Q^{(n)}$ in (2.13), define (σ_n) by*

$$\sigma_n(w) := \operatorname{supp} Q^{(n)}(w,\cdot), \quad w \in W \tag{2.33}$$

where $\operatorname{supp}\mu$ denotes the support of a measure μ. Assume that W is compact. Then the following holds.

(i) *The Markov chain associated with the RSCC is regular if and only if there exists a point $w_0 \in W$ such that*

$$\lim_{n\to\infty} \operatorname{dist}(\sigma_n(w), w_0) = 0 \quad \text{for all } w \in W \tag{2.34}$$

where $\operatorname{dist}(A,w) := \inf_{w'\in A}|w'-w|$ for $A \subset W$.

(ii) *For (σ_n) in (2.33), the following holds:*

$$\sigma_{m+n}(w) = \overline{\bigcup_{w'\in\sigma_m(w)} \sigma_n(w')}, \tag{2.35}$$

for all m, $n \in \mathbb{N}_+$, $w \in W$, where the overline means the topological closure in W.

Proof. (i) See Theorem 3.3.31, p. 116, of [9].

(ii) By assumption, W is a compact metric subspace of $\mathbb{R}$. Hence the Q in (2.10) is "continuous" in the sense of Definition 3.3.1 in [9]. Therefore we can apply Lemma 3.3.32, p. 117 of [9] and the statement holds. □

Definition 2.12. Let $\{(W,\mathcal{W}),X,u,P\}$ be an RSCC and let P_r be as in (2.7).

(i) For $w \in W$, $n,r \in \mathbb{N}_+$, and $A \subset X^r$, define

$$P_r^n(w,A) := P_{r+n-1}(w,\, X^{n-1}\times A), \tag{2.36}$$

with the convention $X^0 \times A := A$.

(ii) ([9], Definition 2.1.4) An RSCC $\{(W,\mathcal{W}),X,u,P\}$ is said to be uniformly ergodic if for any $r \in \mathbb{N}_+$, there exists a probability measure P_r^∞ on $(X^r,\mathcal{P}(X^r))$ such that the sequence $\{P_r^n(w,A) : n \geq 1\}$ in (2.36) converges uniformly to $P_r^\infty(A)$, that is, the following sequence $(\varepsilon_n)_{n\in\mathbb{N}_+}$ goes to 0 when $n \to \infty$:

$$\varepsilon_n := \sup\{|P_r^n(w,A) - P_r^\infty(A)| : w \in W,\, r \in \mathbb{N}_+,\, A \subset X^r\}. \tag{2.37}$$

Theorem 2.13. *Under the Assumption (B), let $\{(W,\mathcal{W}),X,u,P\}$ be an RSCC with contraction such that W is compact. Assume that $\{(W,\mathcal{W}),X,u,P\}$ has a regular associated Markov chain. Then the following holds:*

(i) *$\{(W,\mathcal{W}),X,u,P\}$ is uniformly ergodic.*

(ii) *Let Q^∞ be as in Remark 2.9(iii). Then the probability measure P_r^∞ in (2.37) is given as follows:*

$$P_r^\infty(A) = \int_W P_r(w,A)\, dQ^\infty(w), \quad A \in \mathcal{P}(X^r) \tag{2.38}$$

where P_r is as in (2.7).

Proof. (i) From Theorem 3.4.5, p. 125 in [9], the statement holds.

(ii) By (i), conditions in Remark 2.10(ii) and (iii) are satisfied by assumption. Hence we obtain Q^{∞} on $(W, \mathcal{W})$ in (2.29). From Theorem 3.4.5, p. 125 in [9], the statement holds. □

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. General results in Section 2 will be applied to N-continued fraction expansions.

3.1. RSCC associated with N-continued fraction expansion

Fix an integer $N \geq 1$. We introduce a random system with complete connections (= RSCC) $\{(I, \mathcal{B}_I), \Lambda, u, P\}$ as follows:

$$\begin{cases} u : I \times \Lambda \to I; & u(x,i) := u_{N,i}(x) = \dfrac{N}{x+i}, \\ P : I \times \Lambda \to I; & P(x,i) := V_{N,i}(x) = \dfrac{x+N}{(x+i)(x+i+1)}, \\ \Lambda = \{N, N+1, \ldots\}. \end{cases} \tag{3.1}$$

By the definition of P and using the partial fraction decomposition, $\sum_{i\in\Lambda} P(x,i) = 1$. By (2.10), $Q(x,B) = \sum_{i\in X(x,B)} V_{N,i}(x)$ for $(x,B) \in I \times \mathcal{B}_I$ where $X(x,B) := \{i \in \Lambda : u_{N,i}(x) \in B\}$. Let U and $Q^{(n)}$ be as in (2.21) and (2.14), respectively. By definition, $\{(I, \mathcal{B}_I), \Lambda, u, P\}$ satisfies the Assumption (B) in Section 2.4.

For the dynamical system $(I, \mathcal{B}_I, T_N)$ in Section 1 and a given probability measure μ on $(I, \mathcal{B}_I)$, the ergodic behavior of the RSCC in (3.1) allows us to find the limiting Gauss–Kuzmin distribution F with respect to (T_N, μ):

$$F(x) := \lim_{n\to\infty} \mu(T_N^n < x), \quad x \in I \tag{3.2}$$

and the invariant measure induced by F.

Lemma 3.1. *$\{(I, \mathcal{B}_I), \Lambda, u, P\}$ is an RSCC with contraction.*

Proof. We have

$$\begin{aligned} \frac{d}{dx}u(x,i) &= \frac{d}{dx}u_{N,i}(x) = -\frac{N}{(x+i)^2}, \\ \frac{d}{dx}P(x,i) &= \frac{i^2 + i - 2Ni - (x^2 + 2Nx + N)}{(x+i)^2(x+i+1)^2} \end{aligned} \tag{3.3}$$

for any $x \in I$ and $i \in \Lambda$. Thus,

$$\sup_{x\in I} \left| \frac{d}{dx}u(x,i) \right| \leq \frac{N}{i^2}, \quad i \in \Lambda \tag{3.4}$$

$$\sup_{x\in I} \left| \frac{d}{dx}P(x,i) \right| < \infty. \tag{3.5}$$

Hence the requirements of Definition 2.10 are fulfilled. □

Lemma 3.2. *$\{(I, \mathcal{B}_I), \Lambda, u, P\}$ has a regular associated Markov chain.*

Proof. By Theorem 2.11(i), it is equivalent to that

$$(\Gamma): \quad \left\{ \begin{array}{l} \text{there exists a point } x^* \in I \text{ such that} \\ \lim\limits_{n\to\infty} \operatorname{dist}(\sigma_n(x), x^*) = 0 \quad \text{for all } x \in I \end{array} \right\}$$

where we remark that $W = I = [0, 1]$ in this case. Hence we show the condition (Γ) as follows.

Fix $x \in I$. Let us define the sequence $(x_n)_{n\geq 0}$ in I, recursively by $x_0 := x$, $x_{n+1} := \dfrac{N}{x_n + N}$ for $n \geq 1$. Clearly $x_{n+1} \in \sigma_1(x_n)$ and therefore Theorem 2.11(ii) and an induction argument lead us to the conclusion that $x_n \in \sigma_n(x)$ for $n \in \mathbb{N}_+$. But, $\lim\limits_{n\to\infty} x_n = x^* = \dfrac{-N + \sqrt{N^2 + 4N}}{2}$ for any $x \in I$. Hence $\operatorname{dist}(\sigma_n(x), x^*) \leq |x_n - x^*| \to 0$ as $n \to \infty$. Hence we find $x^* := \dfrac{-N + \sqrt{N^2 + 4N}}{2}$ in the condition (Γ). □

Proposition 3.3. *$\{(I, \mathcal{B}_I), \Lambda, u, P\}$ is uniformly ergodic.*

Proof. In order to apply Theorem 2.13 to the RSCC $\{(I, \mathcal{B}_I), \Lambda, u, P\}$ in (3.1), we check assumptions in Theorem 2.13. By definition, I is compact. By Lemma 3.1, $\{(I, \mathcal{B}_I), \Lambda, u, P\}$ is an RSCC with contraction. By Lemma 3.2, $\{(I, \mathcal{B}_I), \Lambda, u, P\}$ is regular. Hence all assumptions in Theorem 2.13 are satisfied. Hence the statement holds. □

Let $L(I)$ denote the Banach space of all complex-valued Lipschitz continuous functions on I. The regularity of U in (1.4) with respect to $L(I)$ follows from Theorem 2.11. Moreover, the sequence $\{Q^{(n)}(\cdot, \cdot) : n \geq 1\}$ in (2.14) converges uniformly to a probability measure $Q^{(\infty)}$ on $(I, \mathcal{B}_I)$ and that there exist two positive constants $q < 1$ and k such that

$$\|U_n f - U_\infty f\|_L \leq k q^n \|f\|_L, \quad n \in \mathbb{N}_+,\ f \in L(I) \tag{3.6}$$

where

$$U_n : L(I) \to L(I); \quad \{U_n f\}(x) = \int_I f(y)\, dQ_x^{(n)}(y), \tag{3.7}$$

$$U_\infty : L(I) \to \mathbb{C}; \quad U_\infty f = \int_I f(y)\, dQ^{(\infty)}(y) \tag{3.8}$$

with $Q_x^{(n)}(B) := Q^{(n)}(x, B)$ in (2.14).

Proposition 3.4. *The probability $Q^{(\infty)}$ is the invariant probability measure of the transformation T_N.*

Proof. For G_N in Section 1 and Q in (2.10), and on account of the uniqueness of $Q^{(\infty)}$ we have to show that

$$\int_0^1 Q(x, B)\, dG_N(x) = G_N(B), \quad B \in \mathcal{B}_I. \tag{3.9}$$

Since the intervals $[0, u) \subset [0, 1)$ generate $\mathcal{B}_I$, it is sufficient to show the equation (3.9) just for $B = [0, u)$, $0 < u \leq 1$. Let $E(x, N) = \left\lfloor \frac{N}{u} - x \right\rfloor + 1$. Since $Q(x, B) = \sum_{i\in X(x,B)} V_{N,i}(x)$ for $(x, B) \in I \times \mathcal{B}_I$ where $X(x, B) := \{i \in \mathbb{N} : u_{N,i}(x) \in B\}$, we have

$$Q_m(x,[0,u)) = \sum_{\{i\in\mathbb{N}:0\le u_i(x)<u\}} V_{N,i}(x) = \sum_{i\ge E(x,N)} V_{N,i}(x)$$
$$= \frac{N-E(x,N)}{x+E(x,N)}. \tag{3.10}$$

Thus,

$$\int_0^1 Q(x,[0,u))dG_N(x) = \frac{1}{\log\{(N+1)/N\}}\log\frac{x+N}{N} = G_N([0,u)). \tag{3.11}$$

Hence the statement holds. □

3.2. Proof of Theorem 1.1

By using Proposition 3.3, we prove Theorem 1.1.

Proof of Theorem 1.1. Fix an integer $N \ge 1$ and let G_N be as in Section 1. By (1.5), we have

$$\mu\left(T_N^{-n}(A)\right) = \int_A \{U^n f_0\}(x)dG_N(x) \quad \text{for any } n\in\mathbb{N}_+,\ A\in\mathcal{B}_I \tag{3.12}$$

where $f_0(x) = (x+N)(d\mu/d\lambda)(x)$ for $x\in I$. If $d\mu/d\lambda \in L(I)$, then $f_0\in L(I)$ and by (3.8) we have

$$U^{(\infty)}f_0 = \int_I f_0(x)\,Q^{(\infty)}(dx) = \int_I f_0(x)\,G_N(dx)$$
$$= \int_I (d\mu/d\lambda)(x)\,dx = \frac{1}{\log\{(N+1)/N\}}\log\frac{x+N}{N}. \tag{3.13}$$

Taking into account (3.6), there exist two constants $q<1$ and k such that

$$\left\|U^n f_0 - U^{(\infty)}f_0\right\|_L \le kq^n\left\|f_0\right\|_L, \quad n\in\mathbb{N}_+. \tag{3.14}$$

Furthermore, consider the Banach space $C(I)$ of all real-valued continuous functions on I with the norm $\|f\| := \sup_{x\in I}|f(x)|$. Since $L(I)$ is a dense subspace of $C(I)$ we have

$$\lim_{n\to\infty}\left\|\left(U^n - U^{(\infty)}\right)f\right\| = 0 \quad \text{for all } f\in C(I). \tag{3.15}$$

Therefore, (3.15) is valid for a measurable function f_0 which is $Q^{(\infty)}$-almost surely continuous, that is, for a Riemann-integrable function f. Thus, we have

$$\lim_{n\to\infty}\mu\left(G_N^n < x\right) = \lim_{n\to\infty}\int_0^x \{U^n f_0\}(u)G_N(du) \tag{3.16}$$

$$= \frac{1}{\log\{(N+1)/N\}}\log\frac{x+N}{N}\int_0^x G_N(du) \tag{3.17}$$

$$= \frac{1}{\log\{(N+1)/N\}}\log\frac{x+N}{N}. \tag{3.18}$$

Hence (1.6) is proved. □

Acknowledgments

The author would like to express their sincere thanks to Professor Marius Iosifescu for their valuable comments.

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Publ. Math. Debrecen
91/3-4 (2017), 281–295
DOI: 10.5486/PMD.2017.7543

A Gauss–Kuzmin-type theorem for θ-expansions

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Abstract. A generalization of the regular continued fractions was given by S. Chakraborty and B. V. Rao in 2004. For the transformation which generates this expansion and its invariant measure, the Perron–Frobenius operator is given and studied. For this expansion, we apply the method of Szüsz from 1961 and obtain the solution of its Gauss–Kuzmin-type theorem.

1. Introduction

S. Chakraborty and B. V. Rao [3] considered a continued fraction expansion of a number in terms of an irrational $\theta \in (0, 1)$. This new expansion of positive reals, different from the regular continued fraction expansion, is called the θ-expansion.

The purpose of this paper is to give some ergodic properties, and to solve a Gauss–Kuzmin problem for θ-expansions. In order to solve the Gauss–Kuzmin problem, we apply the method of Szüsz [14], [21]. First, we outline the historical framework of this problem. In Section 1.2, we present the current framework. In Section 1.3, we review known results. In Section 1.4, the main theorem will be shown.

Mathematics Subject Classification: 11J70, 11K50, 37C30.
Key words and phrases: continued fractions, invariant measure, Gauss–Kuzmin problem.

1.1. Gauss' Problem and its progress. Any irrational $0 \leq x < 1$ can be written as the infinite regular continued fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} := [0; a_1, a_2, a_3, \ldots], \tag{1.1}$$

where $a_n \in \mathbb{N}_+ := \{1, 2, 3, \ldots\}$ [7]. Define the *regular continued fraction* (or *Gauss*) *transformation* τ on the unit interval $I := [0, 1]$ by

$$\tau(x) = \begin{cases} \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \tag{1.2}$$

where $\lfloor \cdot \rfloor$ denotes the floor (or entire) function. With respect to the asymptotic behavior of iterations $\tau^n = \tau \circ \cdots \circ \tau$ (with τ repeated n times) of τ, in 1800 Gauss wrote (in modern notation) that

$$\lim_{n \to \infty} \lambda\left(\tau^n \leq x\right) = \frac{\log(1+x)}{\log 2}, \quad x \in I, \tag{1.3}$$

where λ denotes the Lebesgue measure on I. In 1812, Gauss asked Laplace [2] to estimate the *n-th error term* $e_n(x)$ defined by

$$e_n(x) := \lambda(\tau^{-n}[0, x]) - \frac{\log(1+x)}{\log 2}, \quad n \geq 1,\ x \in I. \tag{1.4}$$

This has been called *Gauss' Problem*. In 1928, Kuzmin [9] showed that $e_n(x) = \mathcal{O}(q^{\sqrt{n}})$ as $n \to \infty$, uniformly in x with some (unspecified) $0 < q < 1$. Independently, Lévy [12] proved in 1929 that $|e_n(x)| \leq q^n$ for $n \in \mathbb{N}_+$, $x \in I$, with $q = 0.67157\ldots$. For such historical reasons, the *Gauss–Kuzmin–Lévy theorem* is regarded as the first basic result in the rich metrical theory of continued fractions.

Apart from the regular continued fraction expansion, very many other continued fraction expansions were studied [14], [16]. By such a development, generalizations of these problems for non-regular continued fractions are also called as the *Gauss–Kuzmin problem* and the *Gauss–Kuzmin–Lévy problem* [8], [10], [11], [17], [18], [19], [20].

1.2. θ-expansions as dynamical system. In this paper, we consider a generalization of the Gauss transformation, and prove an analogous result.

This transformation was studied by S. Chakraborty and B. V. Rao, and P. S. Chakraborty and A. Dasgupta in [3] and [4], respectively, and by Sebe and Lascu in [20].

Fix an irrational $\theta \in (0,1)$. In [3], the authors showed that any $x \in (0,\theta)$ can be written in the form

$$x = \cfrac{1}{a_1\theta + \cfrac{1}{a_2\theta + \cfrac{1}{a_3\theta + \ddots}}} := [0; a_1\theta, a_2\theta, \ldots], \tag{1.5}$$

where a_n's are non-negative integers. We will simply write (1.5)

$$x := [a_1\theta, a_2\theta, \ldots]. \tag{1.6}$$

Such a_n's are called *incomplete quotients* (or *continued fraction digits*) of x with respect to the expansion in (1.5) in this paper.

This continued fraction is treated as the following dynamical systems.

Definition 1.1. Fix an irrational $\theta \in (0,1)$ and $m \in \mathbb{N}_+$ such that $\theta^2 = 1/m$.

(i) The measure-theoretical dynamical system $([0,\theta], \mathcal{B}_{[0,\theta]}, T_\theta)$ is defined as follows: $\mathcal{B}_{[0,\theta]}$ denotes the σ-algebra of all Borel subsets of $[0,\theta]$, and T_θ is the transformation

$$T_\theta : [0,\theta] \to [0,\theta]; \qquad T_\theta(x) := \begin{cases} \dfrac{1}{x} - \theta \left\lfloor \dfrac{1}{x\theta} \right\rfloor & \text{if } \ x \in (0,\theta], \\ 0 & \text{if } \ x = 0. \end{cases} \tag{1.7}$$

(ii) In addition to (i), we write $([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta, T_\theta)$ as $([0,\theta], \mathcal{B}_{[0,\theta]}, T_\theta)$, with the following probability measure γ_θ on $([0,\theta], \mathcal{B}_{[0,\theta]})$:

$$\gamma_\theta(A) := \frac{1}{\log(1+\theta^2)} \int_A \frac{\theta dx}{1+\theta x}, \quad A \in \mathcal{B}_{[0,\theta]}. \tag{1.8}$$

Define the *quantized index map* $\eta : [0,\theta] \to \mathbb{N}$ by

$$\eta(x) := \begin{cases} \left\lfloor \dfrac{1}{x\theta} \right\rfloor & \text{if } \ x \neq 0, \\ \infty & \text{if } \ x = 0. \end{cases} \tag{1.9}$$

By using T_θ and η, the sequence $(a_n)_{n\in\mathbb{N}_+}$ in (1.5) is obtained as follows:

$$a_n(x) = \eta(T_\theta^{n-1}(x)), \quad n \geq 1, \tag{1.10}$$

with $T_\theta^0(x) = x$.

In this way, T_θ algorithmically generates the θ-expansion.

Proposition 1.1. *Let $([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta, T_\theta)$ be as in Definition 1.1* (ii).

(i) *$([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta, T_\theta)$ is ergodic.*

(ii) *The measure γ_θ is invariant under T_θ, that is, $\gamma_\theta(A) = \gamma_\theta(T_\theta^{-1}(A))$ for any $A \in \mathcal{B}_{[0,\theta]}$.*

Proof. See Section 8 in [3]. □

By Proposition 1.1 (ii), $([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta, T_\theta)$ is a "dynamical system" in the sense of [1, Definition 3.1.3].

1.3. Known results and applications. For θ-expansions, we show known results and their applications in this subsection.

Let $0 < \theta < 1$ and $m \in \mathbb{N}_+$ such that $\theta^2 = 1/m$. In what follows, the stated identities hold for all n in case x has an infinite θ-expansion, and they hold for $n \leq k$ in case x has a finite θ-expansion terminating at the k-th stage [3].

To this end, define real functions $p_n(x)$ and $q_n(x)$, for $n \in \mathbb{N}_+$, by

$$p_n(x) := a_n(x)\theta p_{n-1}(x) + p_{n-2}(x), \tag{1.11}$$

$$q_n(x) := a_n(x)\theta q_{n-1}(x) + q_{n-2}(x), \tag{1.12}$$

with $p_{-1}(x) := 1$, $p_0(x) := 0$, $q_{-1}(x) := 0$ and $q_0(x) := 1$. By using (1.11) and (1.12), we can verify that

$$x = \frac{p_n(x) + T_\theta^n(x)p_{n-1}(x)}{q_n(x) + T_\theta^n(x)q_{n-1}(x)}, \quad n \geq 0, \tag{1.13}$$

and

$$x - \frac{p_n(x)}{q_n(x)} = \frac{(-1)^{n+1}T_\theta^n(x)}{q_n(x)(q_n(x) + T_\theta^n(x)q_{n-1}(x))}, \quad n \geq 0. \tag{1.14}$$

Putting $\mathbb{N}_m := \{m, m+1, \ldots\}$, $m \in \mathbb{N}_+$, the incomplete quotients a_n, $n \in \mathbb{N}_+$, take positive integer values in $\mathbb{N}_m$.

We now introduce a partition of the interval $[0,\theta]$ which is natural to the θ-expansions. Such a partition is generated by the *fundamental intervals* (or

cylinders) of rank n. For any $n \in \mathbb{N}_+$ and $i^{(n)} = (i_1, \ldots, i_n) \in \mathbb{N}_m^n$, define the *fundamental interval associated with* $i^{(n)}$ by

$$I\left(i^{(n)}\right) = \{x \in [0,\theta] : a_k(x) = i_k \text{ for } k = 1, \ldots, n\}, \tag{1.15}$$

where $I\left(i^{(0)}\right) = [0,\theta]$.

Using the ergodicity of T_θ and Birkhoff's ergodic theorem [5], a number of results were obtained.

For q_n in (1.11) it was shown that its asymptotic growth rate β is defined as

$$\beta = \lim_{n\to\infty} \frac{1}{n} \log q_n. \tag{1.16}$$

This is a Lévy-type result. Although the calculation algorithm is correct, Chakraborty and Rao [3] misspelled the expression of β. They should write that for almost all $x \in [0,\theta)$

$$\beta = \frac{-1}{\log(1+\theta^2)} \int_0^\theta \frac{\theta \log x}{1+\theta x} \mathrm{d}x. \tag{1.17}$$

They also give a Khintchin-type result, i.e., the asymptotic value of the arithmetic mean of $a_1, a_2, \ldots, a_n$, where a_n's are given in (1.10). We have

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \infty. \tag{1.18}$$

In [20], Sebe and Lascu proved a Gauss–Kuzmin theorem for the transformation T_θ. In order to solve the problem, they applied the theory of random systems with complete connections (RSCC) by Iosifescu [6]. We recall that a random system with complete connections is a quadruple

$$\left\{\left([0,\theta], \mathcal{B}_{[0,\theta]}\right), \left(\mathbb{N}_m, \mathcal{P}(\mathbb{N}_m)\right), u, P\right\}, \tag{1.19}$$

where $u : [0,\theta] \to [0,\theta]$,

$$u(x) = u_i(x) := \frac{1}{i\theta + x}, \tag{1.20}$$

and P is a transition probability function from $\left([0,\theta], \mathcal{B}_{[0,\theta]}\right)$ to $(\mathbb{N}_m, \mathcal{P}(\mathbb{N}_m))$ given by

$$P(x) = P_i(x) := \frac{\theta x + 1}{(x + i\theta)(x + (i+1)\theta)}. \tag{1.21}$$

Here $\mathcal{P}(\mathbb{N}_m)$ denotes the power set of $\mathbb{N}_m$. Also, the associated Markov operator of RSCC (1.19) is denoted by U and has the transition probability function

$$Q(x, A) = \sum_{i \in W(x,A)} P_i(x), \quad x \in [0,\theta],\ A \in \mathcal{B}_{[0,\theta]}, \tag{1.22}$$

where $W(x,A)=\{i\in\mathbb{N}_m : u_i(x)\in A\}$.

Using the asymptotic and ergodic properties of operators associated with RSCC (1.19), i.e., the ergodicity of RSCC, they obtained a convergence rate result for the Gauss–Kuzmin-type theorem.

For more details about using RSCC in solving Gauss–Kuzmin–Lévy-type theorems, see [10], [17], [18], [19], [20].

1.4. Main theorem. We show our main theorem in this subsection. We mention that applying the Szüsz method, we obtain a better rate of convergence than that obtained in [20].

If $\theta\in(0,1)$ and $m\in\mathbb{N}_+$ such that $\theta^2=1/m$, the measure γ_θ in (1.8) is the unique absolutely continuous invariant measure for the map T_θ in (1.7). In particular, if one iterates any other absolutely continuous invariant measure repeatedly by T_θ, it will converge exponentially to γ_θ.

Let μ be a non-atomic probability measure on $\mathcal{B}_{[0,\theta]}$, and define

$$F_n(x):=\mu(T_\theta^n\le x),\quad x\in[0,\theta],\ n\in\mathbb{N}_+, \tag{1.23}$$

$$F(x):=\lim_{n\to\infty}F_n(x),\quad x\in I, \tag{1.24}$$

with $F_0(x)=\mu([0,x])$.

Then the following holds.

Theorem 1.1 (A Gauss–Kuzmin-type theorem)**.** *Let T_θ and F_n be as in* (1.7) *and* (1.23)*, respectively. Then there exists a constant $0<q<\theta$ such that F_n can be written as*

$$F_n(x)=\frac{\log(1+\theta x)}{\log(1+\theta^2)}+\mathcal{O}(q^n) \tag{1.25}$$

uniformly with respect to $x\in[0,\theta]$.

Remark 1.1. From (1.25), we see that

$$F(x)=\gamma_\theta([0,x]). \tag{1.26}$$

In fact, the Gauss–Kuzmin theorem estimates the error

$$e_\theta(x):=e_\theta(x,\mu)=\mu(T_\theta^n\le x)-\gamma_\theta([0,x]),\quad x\in[0,\theta]. \tag{1.27}$$

The rest of the paper is organized as follows. In Section 2, we derive the associated Perron–Frobenius operator under different probability measures on $([0,\theta],\mathcal{B}_{[0,\theta]})$. We treat the Perron–Frobenius operator of $([0,\theta],\mathcal{B}_{[0,\theta]},\gamma_\theta,T_\theta)$,

and derive its asymptotic behavior. In Section 3, we prove Theorem 1.1 for θ-expansions. In Section 3.1, we give the necessary results used to prove the Gauss–Kuzmin theorem. The essential argument of the proof is the Gauss–Kuzmin-type equation. Using some properties of the Perron–Frobenius operator of T_θ under γ_θ, we give some results concerning the behavior of the derivative of $\{F_n\}$ in (1.23), which will allow us to complete the proof of Theorem 1.1 in Section 3.2.

2. The Perron–Frobenius operator of T_θ under γ_θ

Let $([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta, T_\theta)$ be as in Definition 1.1. In this section, we derive its Perron–Frobenius operator.

Let μ be a probability measure on $([0,\theta], \mathcal{B}_{[0,\theta]})$ such that $\mu((T_\theta)^{-1}(A)) = 0$, whenever $\mu(A) = 0$ for $A \in \mathcal{B}_{[0,\theta]}$. Since μ is non-atomic and T_θ is ergodic, For example, this condition is satisfied if T_θ is μ-preserving, that is, $\mu(T_\theta)^{-1} = \mu$. Let $L^1([0,\theta],\mu) := \{f : [0,\theta] \to \mathbb{C} : \int_0^\theta |f| \mathrm{d}\mu < \infty\}$. The *Perron–Frobenius operator* U of $([0,\theta], \mathcal{B}_{[0,\theta]}, \mu, T_\theta)$ is defined as the bounded linear operator on the Banach space $L^1([0,\theta],\mu)$ such that the following holds:

$$\int_A Uf \,\mathrm{d}\mu = \int_{(T_\theta)^{-1}(A)} f \,\mathrm{d}\mu \quad \text{for all} \quad A \in \mathcal{B}_{[0,\theta]},\ f \in L^1([0,\theta],\mu). \tag{2.1}$$

For more details, see [1], [7] or Appendix A in [10].

Proposition 2.1. *Let $([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta, T_\theta)$ be as in Definition 1.1, and let U denote its Perron–Frobenius operator.*

(i) *The following equation holds:*

$$Uf(x) = \sum_{i \geq m} P_i(x)\, f(u_i(x)), \quad m \in \mathbb{N}_+,\ f \in L^1([0,\theta],\gamma_\theta), \tag{2.2}$$

where P_i and u_i, $i \geq m$, are as in (1.21) *and* (1.20)*, respectively.*

(ii) *Let μ be a probability measure on $\mathcal{B}_{[0,\theta]}$. Assume that μ is absolutely continuous with respect to the Lebesgue measure λ_θ (and denote $\mu \ll \lambda_\theta$, i.e., if $\mu(A) = 0$ for every set A with $\lambda_\theta(A) = 0$), and let $h = d\mu/d\lambda_\theta$ a.e. in $[0,\theta]$. Then the following holds:*

(a) *The Perron–Frobenius operator S of T_θ under μ is given* a.e. *in $[0,\theta]$ by the equation*

$$Sf(x) = \frac{1}{h(x)} \sum_{i \geq m} \frac{1}{(i\theta + x)} f(u_i(x)) h(u_i(x)) \tag{2.3}$$

$$Sf(x) = \frac{Ug(x)}{(1+\theta x)h(x)}, \quad f \in L^1([0,\theta],\mu), \tag{2.4}$$

where $g(x) := (1+\theta x)f(x)h(x)$, $x \in [0,\theta]$. *In addition, the* n*-th power* S^n *of* S *is given as follows:*

$$S^n f(x) = \frac{U^n g(x)}{(1+\theta x)h(x)}, \tag{2.5}$$

for any $f \in L^1([0,\theta],\mu)$ *and any* $n \in \mathbb{N}_+$.

(b) *The Perron–Frobenius operator* V *of* T_θ *under* λ_θ *is given a.e. in* $[0,\theta]$ *by the equation*

$$Vf(x) = \sum_{i \geq m} \frac{1}{(i\theta+x)^2} f(u_i(x)), \ f \in L^1([0,\theta],\lambda_\theta). \tag{2.6}$$

The powers of V *are given a.e. in* $[0,\theta]$ *by the equation*

$$V^n f(x) = \frac{U^n g(x)}{1+\theta x}, \quad f \in L^1([0,\theta],\lambda_\theta), \ n \in \mathbb{N}_+, \tag{2.7}$$

where $g(x) := (1+\theta x)f(x)$, $x \in [0,\theta]$.

(c) *For any* $n \in \mathbb{N}_+$ *and* $A \in \mathcal{B}_{[0,\theta]}$, *we have*

$$\mu\left((T_\theta)^{-n}(A)\right) = \int_A U^n f(x) \mathrm{d}\gamma_\theta(x), \tag{2.8}$$

where $f(x) := (\log(1+\theta^2))\frac{1+x\theta}{\theta^2} h(x)$, $x \in [0,\theta]$.

Proof. See Appendix. □

For a function $f : [0,\theta] \to \mathbb{C}$, define the *variation* $\mathrm{var}_A f$ of f on a subset A of $[0,\theta]$ by

$$\mathrm{var}_A f := \sup \sum_{i=1}^{k-1} |f(t_{i+1}) - f(t_i)|, \tag{2.9}$$

where the supremum being taken over $t_1 < \cdots < t_k$, $t_i \in A$, $1 \leq i \leq k$, and $k \geq 2$. We write simply $\mathrm{var} f$ for $\mathrm{var}_{[0,\theta]} f$. Let $L^\infty([0,\theta])$ denote the collection of all bounded measurable functions $f : [0,\theta] \to \mathbb{C}$. It is known that $L^\infty([0,\theta]) \subset L^1([0,\theta])$. Let $L([0,\theta])$ denote the Banach space of all complex-valued Lipschitz continuous functions on $[0,\theta]$ with the following norm $\|\cdot\|_L$:

$$\|f\|_L := \sup_{x \in [0,\theta]} |f(x)| + s(f), \tag{2.10}$$

with

$$s(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}, \quad f \in L([0, \theta]). \tag{2.11}$$

In the following proposition, we show that the operator U in (2.2) preserves monotonicity and enjoys a contraction property for Lipschitz continuous functions.

Proposition 2.2. *Let U be as in* (2.2).

(i) *Let $f \in L^\infty([0, \theta])$. Then the following holds:*

(a) *If f is non-decreasing (non-increasing), then Uf is non-increasing (non-decreasing).*

(b) *If f is monotone, then*

$$\operatorname{var}(Uf) \leq k_m \cdot \operatorname{var} f \quad \text{where} \quad k_m := \frac{1}{m+1}. \tag{2.12}$$

(ii) *For any $f \in L([0, \theta])$, we have*

$$s(Uf) \leq q \cdot s(f), \tag{2.13}$$

where

$$q := m \left(\sum_{i \geq m} \left(\frac{m}{i^3(i+1)} + \frac{i+1-m}{i(i+1)^3} \right) \right) \tag{2.14}$$

Proof. See Appendix. □

3. Proof of Theorem 1.1

In this section, we prove our main theorem applying the method of Szüsz [21]. Let $\theta \in (0, 1)$ and $m \in \mathbb{N}_+$ such that $\theta^2 = 1/m$.

3.1. Necessary lemmas. In this subsection, we show some lemmas. First, we show that $\{F_n\}$ in (1.23) satisfy a Gauss–Kuzmin-type equation.

Lemma 3.1. *For functions $\{F_n\}$ in* (1.23), *the following Gauss–Kuzmin-type equation holds:*

$$F_{n+1}(x) = \sum_{i \geq m} \left\{ F_n\left(\frac{1}{i\theta}\right) - F_n\left(\frac{1}{i\theta + x}\right) \right\}, \tag{3.1}$$

for $x \in [0, \theta]$ and $n \in \mathbb{N}$.

PROOF. Let $I_n = \{x \in [0,\theta] : T_\theta^n(x) \le x\}$ and

$$I_{n,i} = \left\{ x \in I_n : \ \frac{1}{i\theta + x} < T_\theta^n(x) < \frac{1}{i\theta} \right\}. \tag{3.2}$$

From (1.7) and (1.10), we see that

$$T_\theta^n(x) = \frac{1}{a_{n+1}\theta + T_\theta^{n+1}(x)}, \quad n \in \mathbb{N}_+. \tag{3.3}$$

From the definition of $I_{n,i}$ and (3.3) it follows that for any $n \in \mathbb{N}$, $I_{n+1} = \bigcup_{i\ge m} I_{n,i}$. Then (3.1) holds because $F_{n+1}(x) = \mu(I_{n+1})$ and

$$\mu(I_{n,i}) = F_n\left(\frac{1}{i\theta}\right) - F_n\left(\frac{1}{i\theta + x}\right). \tag{3.4}$$

□

Remark 3.1. Suppose that F_0' exists everywhere in $[0,\theta]$ and is bounded (μ has bounded density). Then by induction, we have that F_n' exists and it is bounded for any $n \in \mathbb{N}_+$. This allows us to differentiate (3.1) term by term, obtaining

$$F_{n+1}'(x) = \sum_{i\ge m} \frac{1}{(i\theta + x)^2} F_n'\left(\frac{1}{i\theta + x}\right). \tag{3.5}$$

We introduce functions $\{f_n\}$ as follows:

$$f_n(x) := (1 + \theta x)F_n'(x), \quad x \in [0,\theta],\ n \in \mathbb{N}. \tag{3.6}$$

Then (3.5) is

$$f_{n+1}(x) = \sum_{i\ge m} P_i(x) f_n\left(u_i(x)\right), \tag{3.7}$$

where $P_i(x)$ and $u_i(x)$ are given in (1.21) and (1.20), respectively. By Proposition 2.1 (i), we have that $f_{n+1}(x) = Uf_n(x)$.

Lemma 3.2. *For $\{f_n\}$ in (3.6), define $M_n := \max\limits_{x\in[0,\theta]} |f_n'(x)|$. Then*

$$M_{n+1} \le q \cdot M_n, \tag{3.8}$$

where q is the constant in (2.14).

PROOF. Since

$$P_i(x) = \frac{1}{\theta}\left[\frac{1-i\theta^2}{x+i\theta} - \frac{1-(i+1)\theta^2}{x+(i+1)\theta}\right], \tag{3.9}$$

we have

$$f'_{n+1}(x) = \sum_{i\geq m} \frac{1-(i+1)\theta^2}{(x+i\theta)(x+(i+1)\theta)^3} f'_n(\alpha_i) - \sum_{i\geq m} \frac{P_i(x)}{(x+i\theta)^2} f'_n(u_i(x)), \tag{3.10}$$

where $u_{i+1}(x) < \alpha_i < u_i(x)$. Now (3.10) implies

$$M_{n+1} \leq M_n \cdot \max_{x\in[0,\theta]} \left(\sum_{i\geq m} \frac{(i+1)\theta^2 - 1}{(x+i\theta)(x+(i+1)\theta)^3} + \sum_{i\geq m} \frac{P_i(x)}{(x+i\theta)^2} \right). \tag{3.11}$$

We now must calculate the maximum value of the sums in this expression. First, we note that

$$\frac{(i+1)\theta^2 - 1}{(x+i\theta)(x+(i+1)\theta)^3} \leq m^2 \frac{(i+1)\theta^2 - 1}{i(i+1)^3}, \tag{3.12}$$

where we used that $0 \leq x \leq \theta$. Next, let

$$h(x) := \sum_{i\geq m} \frac{P_i(x)}{(x+i\theta)^2}. \tag{3.13}$$

By Proposition 2.1 (i) and Proposition 2.2 (i)(a), we have that function h is decreasing for $x \in [0,\theta]$. Hence, $h(x) \leq h(0)$. This leads to

$$\sum_{i\geq m} \frac{P_i(x)}{(x+i\theta)^2} \leq \cdot \sum_{i\geq m} \frac{1}{i^3(i+1)}. \tag{3.14}$$

The relations (3.11), (3.12) and (3.14) imply (3.8) and that q is as in (2.14). □

3.2. Proof of Theorem 1.1. For $\{F_n\}$ in (1.23), we introduce a function $R_n(x)$ such that

$$F_n(x) = \frac{\log(1+\theta x)}{\log(1+\theta^2)} + R_n(x). \tag{3.15}$$

Because $F_n(0) = 0$ and $F_n(\theta) = 1$, we have $R_n(0) = R_n(\theta) = 0$. To prove Theorem 1.1, we have to show the existence of a constant $0 < q < \theta$ such that

$$R_n(x) = \mathcal{O}(q^n). \tag{3.16}$$

For $\{f_n\}$ in (3.6), if we can show that $f_n(x) = \frac{\theta}{\log(1+\theta^2)} + \mathcal{O}(q^n)$, then its integration will show the equation (1.25).

To demonstrate that $f_n(x)$ has this desired form, it suffices to prove the following lemma.

Lemma 3.3. *For any $x \in [0,\theta]$, there exists a constant $q := q(x)$ with $0 < q < \theta$ such that*

$$f_n'(x) = \mathcal{O}(q^n). \tag{3.17}$$

Proof. Let q be as in (2.14). Using Lemma 3.2, to show (3.17), it is enough to prove that $q < \theta$. Since in the particular cases studied for $m := 10$ we have $\theta = 0.316228$ and $q = 0.0533201$, and for $m := 17$ we have $\theta = 0.242536$ and $q = 0.0305636$, we may assume that $q < \theta$ for any $m \in \mathbb{N}$. □

Appendix A. Proofs of propositions

We prove Propositions 2.1 and 2.2 in this section.

Proof of Proposition 2.1. (i) See Proposition 14 (i) in [20].

(ii)(a) Let $T_{\theta,i}$ denote the restriction of T_θ to the subinterval $I_i := \left(\frac{1}{\theta(i+1)}, \frac{1}{\theta i}\right]$, $i \geq m$, $m \in \mathbb{N}$, that is,

$$T_{\theta,i}(x) = \frac{1}{x} - \theta i, \quad x \in I_i. \tag{A.1}$$

Let $C(A) := T_\theta^{-1}(A)$ and $C_i(A) := (T_{\theta,i})^{-1}(A)$ for $A \in \mathcal{B}_{[0,\theta]}$. Since $C(A) = \bigcup_i C_i(A)$, and $C_i \cap C_j$ is a null set when $i \neq j$, we have

$$\int_{C(A)} f \,\mathrm{d}\gamma_\theta = \sum_{i\geq m} \int_{C_i(A)} f \,\mathrm{d}\gamma_\theta, \quad f \in L^1([0,\theta],\gamma_\theta),\ A \in \mathcal{B}_{[0,\theta]}. \tag{A.2}$$

From (A.2), for any $f \in L^1([0,\theta],\gamma_\theta)$ and $A \in \mathcal{B}_{[0,\theta]}$, we have

$$\begin{aligned}
&\int_{C(A)} f(x)\,\mathrm{d}\mu(x) \\
&\quad = \sum_{i\geq m} \int_{C_i(A)} f(x)\,\mathrm{d}\mu(x) = \sum_{i\geq m} \int_{C_i(A)} f(x)h(x)\,\mathrm{d}x \\
&\quad = \sum_{i\geq m} \int_A \frac{f(u_i(y))\,h(u_i(y))}{(\theta i + y)^2}\,\mathrm{d}y = \int_A \sum_{i\geq m} \frac{f(u_i(x))\,h(u_i(x))}{(\theta i + x)^2}\,\mathrm{d}x.
\end{aligned} \tag{A.3}$$

Since $\mathrm{d}\mu = h\mathrm{d}\lambda_\theta$, (2.3) follows from (A.3).

Now, since $g(x) = (\theta x + 1)f(x)h(x)$, from (2.2) we have

$$Ug(x) = (\theta x + 1) \sum_{i\geq m} \frac{h(u_i(x))}{(\theta i + x)^2}\, f(u_i(x)). \tag{A.4}$$

Now, (2.4) follows immediately from (2.3) and (A.4). Using mathematical induction, (2.5) follows easily.

(ii)(b) The formula (2.6) is a consequence of (2.4) and follows immediately.

(ii)(c) See [20]. □

Proof of Proposition 2.2. (i)(a) We will assume that f is non-decreasing. The proof for non-increasing f will be analogous. Let $x < y$, $x, y \in [0, \theta]$. We have $Uf(y) - Uf(x) = S_1 + S_2$, where

$$S_1 = \sum_{i \geq m} P_i(y) \left(f(u_i(y)) - f(u_i(x)) \right), \tag{A.5}$$

$$S_2 = \sum_{i \geq m} \left(P_i(y) - P_i(x) \right) f(u_i(x)). \tag{A.6}$$

Clearly, $S_1 \leq 0$. Now, since $\sum_{i \geq m} P_i(x) = 1$ for any $x \in [0, \theta]$, we can write

$$S_2 = - \sum_{i \geq m} \left(f(u_m(x)) - f(u_i(x)) \right) \left(P_i(y) - P_i(x) \right). \tag{A.7}$$

It can be seen easily that the functions P_i are increasing for all $i \geq m$. Also, using that $f(u_m(x)) \geq f(u_i(x))$, we have that $S_2 \leq 0$. Thus $Uf(y) - Uf(x) \leq 0$, and the proof is complete.

(i)(b) We will assume that f is non-decreasing. The proof for non-increasing f will be analogous. Then by (a) we have

$$\operatorname{var} Uf = Uf(0) - Uf(\theta) = \sum_{i \geq m} \left(P_i(0) f(u_i(0)) - P_i(\theta) f(u_i(\theta)) \right). \tag{A.8}$$

By calculus, we have

$$\begin{aligned} \operatorname{var} Uf &= \sum_{i \geq m} \left(\frac{m}{i(i+1)} f\left(\frac{1}{\theta i} \right) - \frac{m+1}{(i+1)(i+2)} f\left(\frac{1}{\theta(i+1)} \right) \right) \\ &= \frac{1}{m+1} f(\theta) - \sum_{i \geq m} \frac{1}{(i+1)(i+2)} f\left(\frac{1}{\theta(i+1)} \right) \\ &\leq \frac{1}{m+1} f(\theta) - \sum_{i \geq m} \frac{1}{(i+1)(i+2)} f(0) = \frac{1}{m+1} (f(\theta) - f(0)) = \frac{1}{m+1} \operatorname{var} f. \end{aligned}$$

(ii) For $x \neq y$, $x, y \in [0, \theta]$, we have

$$\begin{aligned} \frac{Uf(y) - Uf(x)}{y - x} &= \sum_{i \geq m} \frac{P_i(y) - P_i(x)}{y - x} f(u_i(x)) \\ &\quad - \sum_{i \geq m} P_i(y) \frac{f(u_i(y)) - f(u_i(x))}{u_i(y) - u_i(x)} \cdot u_i(x) u_i(y). \end{aligned} \tag{A.9}$$

We remark that

$$P_i(x) = \frac{\theta(i+1-m)}{\theta(i+1)+x} + \frac{\theta(m-i)}{\theta i + x}, \quad i \geq m, \tag{A.10}$$

and then

$$\sum_{i\geq m} \frac{P_i(y) - P_i(x)}{y-x} f(u_i(x)) = \sum_{i\geq m} \frac{\theta(i+1-m)}{(y+\theta(i+1))(x+\theta(i+1))} \left(f(u_{i+1}(x)) - f(u_i(x))\right). \tag{A.11}$$

Assume that $x > y$. It then follows from (A.9) and (A.11) that

$$\left|\frac{Uf(y) - Uf(x)}{y-x}\right| \leq s(f) \sum_{i\geq m} \left(\frac{\theta^2(i+1-m)}{(y+\theta i)(y+\theta(i+1))^3} + \frac{P_i(y)}{(y+\theta i)^2}\right). \tag{A.12}$$

Since the sum of the right side of (A.12) is bounded by q (see (3.11)), and since

$$s(Uf) = \sup_{x\neq y} \left|\frac{Uf(y) - Uf(x)}{y-x}\right|, \tag{A.13}$$

the proof is complete. □

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(Received November 20, 2015; revised January 8, 2017)

Journal of Number Theory 195 (2019) 51–71

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On convergence rate in the Gauss–Kuzmin problem for θ-expansions

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ARTICLE INFO

Article history:
Received 21 September 2017
Received in revised form 23 April 2018
Accepted 4 May 2018
Available online 20 June 2018
Communicated by F. Pellarin

Keywords:
θ-expansions
Gauss–Kuzmin-problem
Natural extension
Infinite-order-chain

ABSTRACT

After providing an overview of θ-expansions introduced by Chakraborty and Rao, we focus on the Gauss–Kuzmin problem for this new transformation. Actually, we complete our study on these expansions by proving a two-dimensional Gauss–Kuzmin theorem. More exactly, we obtain such a theorem related to the natural extension of the associated measure-dynamical system. Finally, we derive explicit lower and upper bounds of the error term which provide interesting numerical calculations for the convergence rate involved.

1. Introduction

Chakraborty and Rao [2] have introduced the continued fraction expansion of a number in terms of an irrational $\theta \in (0,1)$. This new expansion of positive reals is called *θ-expansion*. We mention that the case $\theta = 1$ refers to regular continued fraction (RCF)

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https://doi.org/10.1016/j.jnt.2018.05.018
0022-314X/

expansions. The study initiated by Chakraborty and Rao on the analogous transformation of the Gauss map, was completed by Chakraborty and Dasgupta. Actually, in [1] was identified the absolutely continuous invariant probability measure of this new transformation only in the particular case $\theta^2 = 1/m$, $m \in \mathbb{N}_+$.

It is only recently that Sebe and Lascu [13] proved the first Gauss–Kuzmin theorem for θ-expansions, applying the method of random systems with complete connections (RSCC) by Iosifescu and Grigorescu [6]. Following the treatment in the case of the RCF, the Gauss–Kuzmin problem for this new transformation can be approached in terms of the associated Perron–Frobenius operator under the invariant measure induced by the limit distribution function. Moreover, using a Wirsing type approach Sebe [12] obtained a near-optimal solution for the Gauss–Kuzmin problem. The strategy was to restrict the domain of the Perron–Frobenius operator to the Banach space of all functions which have a continuous derivative on $[0, \theta]$.

The aim of this paper is to show a two-dimensional Gauss–Kuzmin theorem for θ-expansions. Note that in the literature there are known similar results for other types of expansions (see [3–5,10,11]).

The paper is organized as follows. In the next section we gather prerequisites needed to prove our results in sections 3 and 4. More exactly, in Section 3 we obtain a Gauss–Kuzmin theorem related to the natural extension [9] of the measure-dynamical system corresponding to these expansions. In Section 4 we try to get close to the optimal convergence rate. Here, the characteristic properties of the Perron–Frobenius operator on the Banach space of functions of bounded variations allows us to derive explicit lower and upper bounds of the error term which provide a more refined estimate of the convergence rate involved. In the last section we conclude by giving numerical calculations.

2. Prerequisites

For a fixed $\theta \in (0, 1)$, Chakraborty and Rao [2] showed that any $x \in (0, \theta)$ can be written in the form

$$x = \cfrac{1}{a_1\theta + \cfrac{1}{a_2\theta + \cfrac{1}{a_3\theta + \ddots}}} := [a_1\theta, a_2\theta, a_3\theta, \ldots] \tag{2.1}$$

which is called the θ-*expansion* of x. Here $a_n \in \mathbb{N}_+ := \{1, 2, 3, \ldots\}$. Such a_n's are called θ-*expansion digits* and they are obtained using the transformation

$$T_\theta : [0, \theta] \to [0, \theta]; \quad T_\theta(x) := \begin{cases} \dfrac{1}{x} - \theta \left\lfloor \dfrac{1}{x\theta} \right\rfloor & \text{if } x \in (0, \theta], \\ 0 & \text{if } x = 0. \end{cases} \tag{2.2}$$

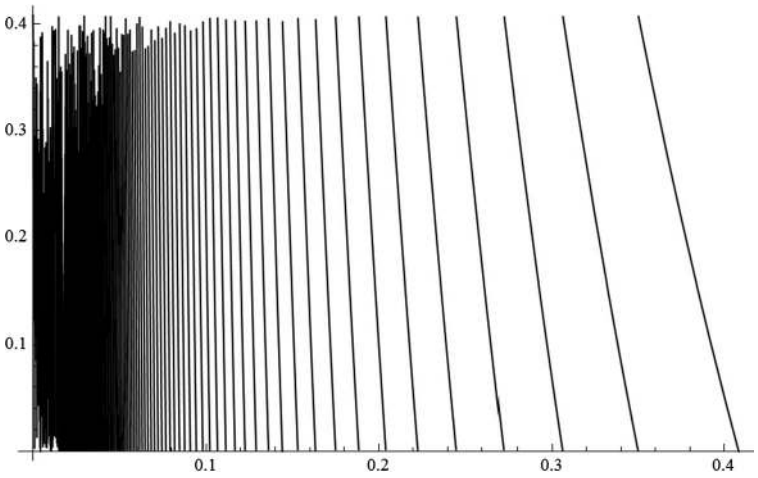

Fig. 1. The transformation T_θ for $\theta = \frac{1}{\sqrt{6}}$.

(See Fig. 1.) Thus, if we define the *quantized index map* $\eta : [0, \theta] \to \mathbb{N} := \{0, 1, 2, \ldots\}$ by

$$\eta(x) := \begin{cases} \left\lfloor \dfrac{1}{x\theta} \right\rfloor & \text{if } x \neq 0, \\ \infty & \text{if } x = 0 \end{cases} \tag{2.3}$$

then the sequence $(a_n)_{n\in\mathbb{N}_+}$ in (2.1) is obtained as follows:

$$a_n(x) = \eta\left(T_\theta^{n-1}(x)\right), \quad n \geq 1, \tag{2.4}$$

with $T_\theta^0(x) = x$.

This new expansion of positive reals, different from the regular continued fraction expansion, was also studied in [1,8,12,13].

In [2] it was shown that θ-expansions are convergent. To this end, define real functions $p_n(x)$ and $q_n(x)$, for $n \in \mathbb{N}_+$, by

$$p_n(x) := a_n(x)\theta p_{n-1}(x) + p_{n-2}(x), \tag{2.5}$$

$$q_n(x) := a_n(x)\theta q_{n-1}(x) + q_{n-2}(x), \tag{2.6}$$

with $p_{-1}(x) := 1$, $p_0(x) := 0$, $q_{-1}(x) := 0$ and $q_0(x) := 1$. It follows that $p_n(x)/q_n(x) = [a_1\theta, a_2\theta, \ldots, a_n\theta]$ which is called the *n-th order convergent* of $x \in [0, \theta]$. One easily shows that for any $x \in [0, \theta]$ it follows

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| \leq \frac{1}{q_n(x)q_{n+1}(x)}, \quad n \in \mathbb{N}_+. \tag{2.7}$$

Then $p_n(x)/q_n(x) \to x$, $n \to \infty$.

In [2], Chakraborty and Rao showed that for $\theta^2 = 1/m$, $m \in \mathbb{N}_+$, T_θ is ergodic with respect to the measure γ_θ defined by

$$\gamma_\theta(A) := \frac{1}{\log(1+\theta^2)} \int_A \frac{\theta dx}{1+\theta x}, \quad A \in \mathcal{B}_{[0,\theta]}, \tag{2.8}$$

where $\mathcal{B}_{[0,\theta]}$ denotes the σ-algebra of all Borel subsets of $[0,\theta]$. Let us note that γ_θ is T_θ-invariant, that is, $\gamma_\theta\left(T_\theta^{-1}(A)\right) = \gamma_\theta(A)$ for any $A \in \mathcal{B}_{[0,\theta]}$. Therefore, $(a_n)_{n\in\mathbb{N}_+}$ is a strictly stationary sequence on $(I, \mathcal{B}_{[0,\theta]}, \gamma_\theta)$ and $a_n \geq m$ for any $m \in \mathbb{N}_+$.

Put $\mathbb{N}_m := \{m, m+1, \ldots\}$, $m \in \mathbb{N}_+$. For any $n \in \mathbb{N}_+$ and $i^{(n)} = (i_1, \ldots, i_n) \in \mathbb{N}_m^n$ we will say that

$$I\left(i^{(n)}\right) = \{x \in [0,\theta] : a_k(x) = i_k \text{ for } k = 1, \ldots, n\} \tag{2.9}$$

is the *n-th order cylinder* and make the convention that $I\left(i^{(0)}\right) = [0,\theta]$. For example, for any $i \in \mathbb{N}_m$ we have

$$I(i) = \{x \in [0,\theta] : a_1(x) = i\} = \left(\frac{1}{(i+1)\theta}, \frac{1}{i\theta}\right). \tag{2.10}$$

2.1. Natural extension, extended random variables and Perron–Frobenius operators

Let $m \in \mathbb{N}_+$ and an irrational $\theta \in (0,1)$ with $\theta^2 = 1/m$. In this section, we introduce the natural extension $\overline{T}_\theta$ of T_θ in (2.2) and its extended random variables according to Chap. 1.3 of [7], and we consider the Perron–Frobenius operator of T_θ.

2.1.1. Natural extension

Let $\left([0,\theta], \mathcal{B}_{[0,\theta]}, T_\theta\right)$ be as above. Define $(u_i)_{i\in\mathbb{N}_m}$ by

$$u_i : [0,\theta] \to [0,\theta]; \quad u_i(x) := \frac{1}{x+i\theta}, \quad x \in [0,\theta]. \tag{2.11}$$

For each $i \in \mathbb{N}_m$, u_i is a right inverse of T_θ, that is,

$$(T_\theta \circ u_i)(x) = x, \quad \text{for any } x \in [0,\theta]. \tag{2.12}$$

Furthermore, if $\eta(x) = i$, then $(u_i \circ T_\theta)(x) = x$ where η is as in (2.3).

Definition 2.1. The *natural extension* $\left([0,\theta]^2, \mathcal{B}^2_{[0,\theta]}, \overline{T}_\theta\right)$ of $\left([0,\theta], \mathcal{B}_{[0,\theta]}, T_\theta\right)$ is the transformation $\overline{T}_\theta$ of the square space $\left([0,\theta]^2, \mathcal{B}^2_{[0,\theta]}\right) := \left([0,\theta], \mathcal{B}_{[0,\theta]}\right) \times \left([0,\theta], \mathcal{B}_{[0,\theta]}\right)$ defined as follows [9]:

$$\overline{T}_\theta : [0,\theta]^2 \to [0,\theta]^2; \quad \overline{T}_\theta(x,y) := \left(T_\theta(x),\, u_{\eta(x)}(y)\right),\ (x,y) \in [0,\theta]^2. \tag{2.13}$$

From (2.12), we see that $\overline{T}_\theta$ is bijective on $[0,\theta]^2$ with the inverse

$$(\overline{T}_\theta)^{-1}(x,y) = \left(u_{\eta(y)}(x),\, T_\theta(y)\right), \quad (x,y) \in [0,\theta]^2. \tag{2.14}$$

Iterations of (2.13) and (2.14) are given as follows for each $n \geq 2$:

$$\left(\overline{T}_\theta\right)^n (x,y) = \left(\, T_\theta^n(x),\, [x_n\theta, x_{n-1}\theta, \ldots, x_2\theta,\, x_1\theta + y]\, \right), \tag{2.15}$$

$$\left(\overline{T}_\theta\right)^{-n} (x,y) = \left(\, [y_n\theta, y_{n-1}\theta, \ldots, y_2\theta,\, y_1\theta + x],\, T_\theta^n(y)\, \right) \tag{2.16}$$

where $x_i := \eta(T_\theta^{i-1}(x))$ and $y_i := \eta(T_\theta^{i-1}(y))$ for $i = 1, \ldots, n$.

For γ_θ in (2.8), define its *extended measure* $\overline{\gamma}_\theta$ on $\left([0,\theta]^2, \mathcal{B}^2_{[0,\theta]}\right)$ as

$$\overline{\gamma}_\theta(B) := \frac{1}{\log(1+\theta^2)} \iint\limits_B \frac{\mathrm{d}x\mathrm{d}y}{(1+xy)^2}, \quad B \in \mathcal{B}^2_{[0,\theta]}. \tag{2.17}$$

Then $\overline{\gamma}_\theta(A \times [0,\theta]) = \overline{\gamma}_\theta([0,\theta] \times A) = \gamma_\theta(A)$ for any $A \in \mathcal{B}_{[0,\theta]}$.

The measure $\overline{\gamma}_\theta$ is preserved by $\overline{T}_\theta$ [13], i.e., $\overline{\gamma}_\theta((\overline{T}_\theta)^{-1}(B)) = \overline{\gamma}_\theta(B)$ for any $B \in \mathcal{B}^2_{[0,\theta]}$. Since $\overline{T}_\theta$ is invertible on $[0,\theta]^2$, the last equation is equivalent to

$$\overline{\gamma}_\theta\left(\overline{T}_\theta(B)\right) = \overline{\gamma}_\theta(B), \quad \text{for any } B \in \mathcal{B}^2_{[0,\theta]}. \tag{2.18}$$

2.1.2. Extended random variables

Define the projection $E : [0,\theta]^2 \to [0,\theta]$ by $E(x,y) := x$. With respect to $\overline{T}_\theta$ in (2.2), define *extended incomplete quotients* $\overline{a}_l(x,y)$, $l \in \mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ at $(x,y) \in [0,\theta]^2$ by

$$\overline{a}_l(x,y) := (\eta \circ E)(\, (\overline{T}_\theta)^{l-1}(x,y)\,), \quad l \in \mathbb{Z}. \tag{2.19}$$

Remark that $\overline{a}_l(x,y)$ in (2.19) is also well-defined for $l \leq 0$ because $\overline{T}_\theta$ is invertible. For any $n \in \mathbb{N}_+$ and $(x,y) \in [0,\theta]^2$, by (2.15) and (2.16), we have

$$\overline{a}_n(x,y) = x_n, \quad \overline{a}_0(x,y) = y_1, \quad \overline{a}_{-n}(x,y) = y_{n+1}, \tag{2.20}$$

where we use notations in (2.15) and (2.16).

Since $\overline{\gamma}_\theta$ is preserved by $\overline{T}_\theta$, the doubly infinite sequence $(\overline{a}_l(x,y))_{l\in\mathbb{Z}}$ is strictly stationary (i.e., its distribution is invariant under a shift of the indices) under $\overline{\gamma}_\theta$. The stochastic property of $(\overline{a}_l)_{l\in\mathbb{Z}}$ follows from the fact that

$$\overline{\gamma}_\theta([0,x] \times [0,\theta] \,|\, \overline{a}_0, \overline{a}_{-1}, \ldots) = \frac{(a\theta+1)x}{(ax+1)\theta} \quad \overline{\gamma}_\theta\text{-a.s.}, \tag{2.21}$$

for any $x \in [0,\theta]$, where $a := [\overline{a}_0\theta, \overline{a}_{-1}\theta, \ldots]$ with $\overline{a}_l := \overline{a}_l(x,y)$ for $l \in \mathbb{Z}$ and $(x,y) \in [0,\theta]^2$.

If I_n denote the n-th order cylinder $I(\overline{a}_0, \overline{a}_{-1}, \ldots, \overline{a}_{-n})$ for $n \in \mathbb{N}$ and $(\overline{a}_1 = i) = I(i) \times [0,\theta]$, $i \in \mathbb{N}_m$, it follows that

$$\begin{aligned}\overline{\gamma}_\theta\left(\overline{a}_1 = i | \, \overline{a}_0, \overline{a}_{-1}, \ldots\right) &= \lim_{n\to\infty} \overline{\gamma}_\theta\left(\overline{a}_1 = i \mid I_n\right) = \lim_{n\to\infty} \overline{\gamma}_\theta\left(I(i) \times [0,\theta] | \, I_n\right) \\ &= \lim_{n\to\infty} \frac{\overline{\gamma}_\theta\left(I(i) \times I_n\right)}{\overline{\gamma}_\theta\left([0,\theta] \times I_n\right)} = \lim_{n\to\infty} \frac{1}{\gamma_\theta(I_n)} \int\limits_{I_n} P_i(y) \mathrm{d}\gamma_\theta(y) \\ &= \lim_{n\to\infty} P_i(y_n) = P_i(a) \quad \overline{\gamma}_\theta\text{-a.s.}, \end{aligned} \tag{2.22}$$

where $y_n \in I_n$ with $\lim\limits_{n\to\infty} y_n = [\overline{a}_0\theta, \overline{a}_{-1}\theta, \ldots] = a$ and

$$P_i(x) := \frac{x\theta + 1}{(x + i\theta)(x + (i+1)\theta)}. \tag{2.23}$$

The strict stationarity of $(\overline{a}_l)_{l\in\mathbb{Z}}$ under $\overline{\gamma}_\theta$ implies that

$$\overline{\gamma}_\theta(\overline{a}_{l+1} = i \mid \overline{a}_l, \overline{a}_{l-1}, \ldots) = P_i(a) \quad \overline{\gamma}_\theta\text{-a.s.} \tag{2.24}$$

for any $i \in \mathbb{N}_m$ and $l \in \mathbb{Z}$.

The last equation emphasizes that $(\overline{a}_l)_{l\in\mathbb{Z}}$ is an infinite-order-chain in the theory of dependence with complete connections (see [6], Section 5.5).

Motivated by (2.21), we shall consider the one-parameter family $\{\gamma_{\theta,a} : a \in [0,\theta]\}$ of (conditional) probability measures on $([0,\theta], \mathcal{B}_{[0,\theta]})$ defined by their distribution functions

$$\gamma_{\theta,a}([0,x]) := \frac{(a\theta+1)x}{(ax+1)\theta}, \quad x \in [0,\theta], \ a \in [0,\theta]. \tag{2.25}$$

Note that $\gamma_{\theta,0}$ is the Lebesgue measure λ_θ on $[0,\theta]$.

Let a_n's be as in (2.4). For each $a \in [0,\theta]$, define $(s_{n,a})_{n\in\mathbb{N}_+}$ by

$$s_{0,a} := a, \quad s_{n,a} := \frac{1}{a_n\theta + s_{n-1,a}}, \quad n \in \mathbb{N}_+.$$

Then we have

$$s_{1,a} = \frac{1}{a_1\theta + a}, \quad s_{n,a} = \left[a_n\theta, \ldots, a_2\theta, a_1\theta + a\right], \, n \geq 2.$$

Note that

$$\gamma_{\theta,a}\left(A | a_1, \ldots, a_n\right) = \gamma_{\theta,s_{n,a}}\left(T_\theta^n(A)\right),$$

for all $a \in [0,\theta]$, $n \in \mathbb{N}_+$ and for any A belonging to the σ-algebra generated by the random variables $a_{n+1}, a_{n+2}, \ldots$. In particular, it follows that for any $a \in [0,\theta]$

$$\gamma_{\theta,a}\left(T_\theta^n < x \middle| a_1, \ldots, a_n\right) = \frac{(s_{n,a}\theta + 1)x}{(s_{n,a}x + 1)\theta} \tag{2.26}$$

for any $x \in [0,\theta]$, $n \in \mathbb{N}_+$.

2.1.3. Perron–Frobenius operator of T_θ

Let $([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta, T_\theta)$ be as in (2.2) and (2.8) and let $L^1([0,\theta],\gamma_\theta) := \{f : [0,\theta] \to \mathbb{C} : \int_0^\theta |f| \mathrm{d}\gamma_\theta < \infty\}$. The *Perron–Frobenius operator* of $([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta, T_\theta)$ is defined as the bounded linear operator U on the Banach space $L^1([0,\theta],\gamma_\theta)$ such that the following holds [13]:

$$Uf(x) = \sum_{i\in\mathbb{N}_m} P_i(x)\, f(u_i(x)), \quad f \in L^1([0,\theta],\gamma_\theta) \tag{2.27}$$

where P_i and u_i are as in (2.23) and (2.11), respectively.

For a function $f : [0,\theta] \to \mathbb{C}$, define the *variation* $\mathrm{var}_A f$ of f on a subset A of $[0,\theta]$ by

$$\mathrm{var}_A f := \sup \sum_{i=1}^{k-1} |f(t_{i+1}) - f(t_i)|, \tag{2.28}$$

where the supremum being taken over $t_1 < \cdots < t_k$, $t_i \in A$, $i = 1, \ldots, k$ and $k \geq 2$. We write simply $\mathrm{var} f$ for $\mathrm{var}_{[0,\theta]} f$. Let $BV([0,\theta]) := \{f : [0,\theta] \to \mathbb{C} : \mathrm{var}\, f < \infty\}$ and let $L^\infty([0,\theta])$ denote the collection of all bounded measurable functions $f : [0,\theta] \to \mathbb{C}$. It is known that $BV([0,\theta]) \subset L^\infty([0,\theta]) \subset L^1([0,\theta],\gamma_\theta)$. In [8] we showed that

$$\mathrm{var}\,(Uf) \leq \frac{1}{m+1}\mathrm{var} f \tag{2.29}$$

where U is as in (2.27) and $f \in BV([0,\theta])$ is a real-valued function.

If $f \in L^\infty([0,\theta])$ define the linear functional U_∞ by

$$U_\infty : L^\infty([0,\theta]) \to \mathbb{C}; \quad U_\infty f = \int_0^\theta f(x)\, \gamma_\theta(\mathrm{d}x). \tag{2.30}$$

Then we have

$$U_\infty U^n f = U_\infty f \quad \text{for any } n \in \mathbb{N}_+. \tag{2.31}$$

Proposition 2.2. *For any $f \in BV([0,\theta])$ and for all $n \in \mathbb{N}_+$ we have*

$$\mathrm{var}\, U^n f \leq \frac{1}{(m+1)^n}\mathrm{var} f, \tag{2.32}$$

$$|U^n f - U_\infty f| \leq \frac{1}{(m+1)^n}\mathrm{var} f. \tag{2.33}$$

Proof. Note that for any $f \in BV([0,\theta])$ and $u \in [0,\theta]$ we have

$$|f(u)| - \left|\int_0^\theta f(x)\gamma_\theta(\mathrm{dx})\right| \le \left|f(u) - \int_0^\theta f(x)\gamma_\theta(\mathrm{dx})\right| = \left|\int_0^\theta (f(u)-f(x))\gamma_\theta(\mathrm{dx})\right| \le \mathrm{var} f,$$

whence

$$|f| \le \left|\int_0^\theta f(x)\gamma_\theta(\mathrm{dx})\right| + \mathrm{var} f, \quad f \in BV([0,\theta]). \tag{2.34}$$

Finally, (2.31) and (2.34) imply that

$$|U^n f - U_\infty f| \le \mathrm{var}\,(U^n f - U_\infty f) = \mathrm{var}\, U^n f$$

for all $n \in \mathbb{N}$ and $f \in BV([0,\theta])$, which leads to (2.33). □

By induction with respect to $n \in \mathbb{N}$ we get

$$U^n f(x) = \sum_{i_1,\dots,i_n \in \mathbb{N}_m} P_{i_1\dots i_n}(x) f(u_{i_n\dots i_1}(x)), \quad x \in [0,\theta]$$

where

$$u_{i_n\dots i_1} = u_{i_n} \circ \dots \circ u_{i_1} \tag{2.35}$$

$$P_{i_1\dots i_n}(x) = P_{i_1}(x) P_{i_2}(u_{i_1}(x)) \dots P_{i_n}(u_{i_{n-1}\dots i_1}(x)), \quad n \ge 2. \tag{2.36}$$

Here the functions u_i and P_i are defined in (2.11) and (2.23), respectively, for all $i \in \mathbb{N}_m$.

Putting

$$\frac{p_n(i_1,\dots,i_n)}{q_n(i_1,\dots,i_n)} = [i_1\theta,\dots,i_n\theta], \quad n \in \mathbb{N}_+,$$

for arbitrary indeterminates $i_1,\dots,i_n$, we get

$$P_{i_1\dots i_n}(a) = \frac{1+a\theta}{q_{n-1}(i_2,\dots,i_n)(a+i_1\theta)+p_{n-1}(i_2,\dots,i_n)} \times \frac{1}{q_n(i_2,\dots,i_n,m)(a+i_1\theta)+p_n(i_2,\dots,i_n,m)} \tag{2.37}$$

for all $n \ge 2$, $i_n \in \mathbb{N}_m$, and $a \in [0,\theta]$.

3. Gauss–Kuzmin theorem related to the natural extension

Let $\left([0,\theta]^2, \mathcal{B}^2_{[0,\theta]}, \overline{\gamma}_\theta, \overline{T}_\theta\right)$ be as in Section 2. In this section a Gauss–Kuzmin theorem for $([0,\theta]^2, \mathcal{B}^2_{[0,\theta]}, \overline{\gamma}_\theta, \overline{T}_\theta)$ is given. First we give a modified version of the Gauss–Kuzmin theorem for T_θ proved in [8]. Then we show some important results used in the proof of the main theorem.

3.1. Gauss–Kuzmin theorems for T_θ and $\overline{T}_\theta$

Theorem 3.1 *(A Gauss–Kuzmin theorem for T_θ). Let $([0,\theta], \mathcal{B}_{[0,\theta]}, \gamma_\theta, T_\theta)$ as in (2.2) and (2.8). There exists a constant $0 < q < \theta$ such that for any $A \in \mathcal{B}_{[0,\theta]}$ we have*

$$\left|\lambda_\theta\left(T_\theta^{-n}(A)\right) - \gamma_\theta(A)\right| < Cq^n\lambda_\theta(A), \tag{3.1}$$

where C is an universal constant.

Proof. In [13] (Prop. 14(ii)) we show that $\mu\left((T_\theta)^{-n}(A)\right) = \int_A U^n f(x)\mathrm{d}\gamma_\theta(x)$ where μ is a probability measure on $\left([0,\theta], \mathcal{B}_{[0,\theta]}\right)$ absolutely continuous with respect to the Lebesgue measure λ_θ, and $f(x) := (\log(1+\theta^2))\frac{1+\theta x}{\theta^2}h(x)$ with $h := d\mu/d\lambda_\theta$ a.e. in $[0,\theta]$. In the special case $\mu = \lambda_\theta$ we obviously have

$$\lambda_\theta\left(T_\theta^{-n}(A)\right) = \frac{\theta}{\log(1+\theta^2)}\int_A \frac{U^n f(x)}{1+\theta x}\mathrm{d}x \tag{3.2}$$

with $f(x) := (\log(1+\theta^2))\frac{1+\theta x}{\theta^2}$, $x \in [0,\theta]$. Thus, from (2.30) we have that $U_\infty f = 1$. Therefore,

$$\gamma_\theta(A) = \frac{\theta}{\log(1+\theta^2)}\int_A \frac{U_\infty f}{1+\theta x}\mathrm{d}x. \tag{3.3}$$

Using [13] (Prop. 27) it follows that there exist two positive constants $q < \theta$ and K such that

$$\left\|U^n f - U_\infty f\right\|_L \le Kq^n \left\|f\right\|_L, \quad f \in L([0,\theta]), n \in \mathbb{N}_+ \tag{3.4}$$

where $L([0,\theta])$ denote the Banach space of all complex-valued Lipschitz continuous functions on $[0,\theta]$ with the following norm:

$$\|f\|_L := \sup_{x\in[0,\theta]}|f(x)| + \sup_{x'\neq x''}\frac{|f(x') - f(x'')|}{|x' - x''|}.$$

Therefore

$$\begin{aligned}\left|\lambda_\theta\left(T_\theta^{-n}(A)\right)-\gamma_\theta(A)\right| &\leq \frac{\theta}{\log\left(1+\theta^2\right)}\int\limits_A \frac{\left|U^n f(x)-U_\infty f\right|}{1+\theta x}\mathrm{d}x\\ &< Kq^n\left\|f\right\|_L \frac{\theta}{\log\left(1+\theta^2\right)}\int\limits_A \frac{1}{1+\theta x}\mathrm{d}x\\ &= Kq^n\left\|f\right\|_L \gamma_\theta(A)\end{aligned}$$

and since

$$\gamma_\theta(A)\leq \frac{\theta}{\log\left(1+\theta^2\right)}\lambda_\theta(A),\quad A\in\mathcal{B}_{[0,\theta]}$$

then the proof is complete. □

In [8] the proof of Gauss–Kuzmin theorem is based on the Gauss–Kuzmin-type equation which in this case is

$$F_{n+1}(x)=\sum_{i\geq m}\left\{F_n\left(\frac{1}{i\theta}\right)-F_n\left(\frac{1}{i\theta+x}\right)\right\} \tag{3.5}$$

where the functions $(F_n)_{n\in\mathbb{N}}$ are defined for $x\in[0,\theta]$ by

$$F_0(x):=\lambda_\theta([0,x])=\frac{x}{\theta},\quad F_n(x):=\lambda_\theta(T_\theta^n\leq x),\ n\in\mathbb{N}_+. \tag{3.6}$$

The measure γ_θ defined in (2.8) is an eigenfunction of (3.5), namely, if we put $F_n(x)=\log(1+\theta x)$, $x\in[0,\theta]$, we obtain $F_{n+1}(x)=\log(1+\theta x)$. The factor $1/\log(1+\theta^2)$ is a normalizing constant.

We give now the main theorem of this section. First, define $\Delta_{x,y}=[0,x]\times[0,y]$ for any $x,y\in[0,\theta]$, and the functions $(\overline{F}_n)_{n\in\mathbb{N}_+}$ on $[0,\theta]^2$ by

$$\overline{F}_n(x,y):=\overline{\lambda}_\theta\left(\left\{(\xi,\zeta)\in[0,\theta]^2:\left(\overline{T}_\theta\right)^n(\xi,\zeta)\in\Delta_{x,y}\right\}\right), \tag{3.7}$$

where $\overline{\lambda}_\theta$ is the Lebesgue measure on $[0,\theta]^2$.

Theorem 3.2 *(A Gauss–Kuzmin theorem for $\overline{T}_\theta$). For every $n\geq 2$ and $(x,y)\in[0,\theta]^2$ one has*

$$\overline{F}_n(x,y)=\frac{\log(1+xy)}{\log(1+\theta^2)}+\mathcal{O}\left(\rho^n\right) \tag{3.8}$$

with $0<\rho<1$.

3.2. Necessary results

In this subsection, we give necessary results used to prove the Gauss–Kuzmin theorem for $\overline{T}_\theta$. Like in the one-dimensional case, we need the Gauss–Kuzmin-type equation associated with the functions $(\overline{F}_n)_{n\in\mathbb{N}_+}$ defined in (3.7). Thus, for any $0 < y \le \theta$, put $\ell_1 := \eta(y)$, where η is as in (2.3). Then $\left(\overline{T}_\theta\right)^{n+1}(x,y) \in \Delta_{x,y}$ is equivalent to

$$\left(\overline{T}_\theta\right)^n \in \left\{ \bigcup_{i\ge\ell_1+1} \left[\frac{1}{i\theta+x}, \frac{1}{i\theta}\right] \times [0,\theta] \right\} \cup \left\{ \left[\frac{1}{\ell_1\theta+x}, \frac{1}{\ell_1\theta}\right] \times \left[\frac{1}{y} - \ell_1\theta, \theta\right] \right\}.$$

From this and (3.7) we get the Gauss–Kuzmin-type equation on $[0,\theta]^2$:

$$\begin{aligned} \overline{F}_{n+1}(x,y) = &\sum_{i\ge\ell_1} \left\{ \overline{F}_n\left(\frac{1}{i\theta}, \theta\right) - \overline{F}_n\left(\frac{1}{i\theta+x}, \theta\right) \right\} \\ &- \left\{ \overline{F}_n\left(\frac{1}{\ell_1\theta}, \frac{1}{y} - \ell_1\theta\right) - \overline{F}_n\left(\frac{1}{\ell_1\theta+x}, \frac{1}{y} - \ell_1\theta\right) \right\}. \end{aligned} \tag{3.9}$$

A straightforward calculation shows that the measure $\overline{\gamma}_\theta$ defined in (2.17) is an eigenfunction of (3.9), namely, if we put $\overline{F}_n(x,y) = \log(1+xy)$, $x, y \in [0,\theta]$, we obtain $\overline{F}_{n+1}(x,y) = \log(1+xy)$.

Lemma 3.3. *Let $n \in \mathbb{N}$, $n \ge 2$ and let $y \in [0,\theta] \cap \mathbb{Q}$ with $y = [\ell_1\theta, \dots, \ell_d\theta]$, $\ell_1, \dots, \ell_d \in \mathbb{N}_m$, $\ell_d \ge m+1$, where $d \le \lfloor n/2 \rfloor$. Then for every $x, x^* \in [0,\theta)$ with $x^* < x$,*

$$\begin{aligned} &\left| \overline{F}_n(x,y) - \overline{F}_n(x^*,y) - \frac{1}{\log(1+\theta^2)} \log\left(\frac{1+xy}{1+x^*y}\right) \right| \\ &\quad < \frac{(1+\theta^2)^2}{\theta^2} \frac{C}{1-q} \overline{\lambda}_\theta(\Delta_{x,y} \setminus \Delta_{x^*,y}) q^{n-d} \end{aligned}$$

where $0 < q < \theta$ and C as given in Theorem 3.1.

Proof. Let $y_0 = y$, $y_i := [\ell_{i+1}\theta, \dots, \ell_d\theta]$, $i = 1, \dots, d$, with $y_d = 0$. Then $y_1 = \frac{1}{y} - \ell_1\theta$. Applying (3.9) one gets

$$\begin{aligned} \overline{F}_n(x,y) - \overline{F}_n(x^*,y) = &\sum_{i\ge\ell_1} \left\{ \overline{F}_{n-1}\left(\frac{1}{i\theta+x^*}, \theta\right) - \overline{F}_{n-1}\left(\frac{1}{i\theta+x}, \theta\right) \right\} \\ &+ \left\{ \overline{F}_{n-1}\left(\frac{1}{\ell_1\theta+x}, y_1\right) - \overline{F}_{n-1}\left(\frac{1}{\ell_1\theta+x^*}, y_1\right) \right\}. \end{aligned} \tag{3.10}$$

Now for each $B \in \mathcal{B}^2_{[0,\theta]}$ one has

$$\frac{1}{(1+\theta^2)}\frac{1}{\log(1+\theta^2)}\overline{\lambda}_\theta(B) \le \overline{\gamma}_\theta(B) \le \frac{1}{\log(1+\theta^2)}\overline{\lambda}_\theta(B). \tag{3.11}$$

Now from (2.10) and (3.11), it follows that:

$$\begin{aligned}
\sum_{i\ge\ell_1}\lambda_\theta\left(\left[\frac{1}{i\theta+x},\frac{1}{i\theta+x^*}\right]\right) &= \sum_{i\ge\ell_1}\frac{1}{\theta}\left(\frac{1}{i\theta+x^*}-\frac{1}{i\theta+x}\right)\\
&= \sum_{i\ge\ell_1}\overline{\lambda}_\theta\left\{\left(\frac{1}{i\theta+x},\frac{1}{i\theta+x^*}\right)\times[0,\theta]\right\}\\
&\le (1+\theta^2)\log\left(1+\theta^2\right)\sum_{i\ge\ell_1}\overline{\gamma}_\theta\left\{\left(\frac{1}{i\theta+x},\frac{1}{i\theta+x^*}\right)\times[0,\theta]\right\}\\
&= (1+\theta^2)\log\left(1+\theta^2\right)\sum_{i\ge\ell_1}\overline{\gamma}_\theta\left\{(x^*,x)\times I(i)\right\}\\
&\le (1+\theta^2)\log\left(1+\theta^2\right)\frac{1}{\log(1+\theta^2)}\sum_{i\ge\ell_1}\overline{\lambda}_\theta\left\{(x^*,x)\times I(i)\right\}\\
&= (1+\theta^2)\sum_{i\ge\ell_1}\overline{\lambda}_\theta\left\{(x^*,x)\times\left(\frac{1}{(i+1)\theta},\frac{1}{i\theta}\right)\right\}\\
&= (1+\theta^2)\frac{x-x^*}{\theta}\sum_{i\ge\ell_1}\lambda_\theta\left\{\left(\frac{1}{(i+1)\theta},\frac{1}{i\theta}\right)\right\}\\
&= (1+\theta^2)\frac{x-x^*}{\theta}\frac{1}{\theta}\sum_{i\ge\ell_1}\left(\frac{1}{i\theta}-\frac{1}{(i+1)\theta}\right)\\
&\le (1+\theta^2)\frac{x-x^*}{\theta^2}\frac{1}{\ell_1\theta} \le (1+\theta^2)\frac{x-x^*}{\theta}(m+1)\frac{y}{\theta}\\
&= \frac{(1+\theta^2)^2}{\theta^2}\overline{\lambda}_\theta\left(\Delta_{x,y}\setminus\Delta_{x^*,y}\right).
\end{aligned} \tag{3.12}$$

For every $2\le k\le d$, a similar analysis leads to

$$\begin{aligned}
&\sum_{i\ge\ell_k}\lambda_\theta\left\{[i\theta,\ell_{k-1}\theta,\dots,\ell_1\theta+x],[i\theta,\ell_{k-1}\theta,\dots,\ell_1\theta+x^*]\right\}\\
&= \frac{1}{\theta}\sum_{i\ge\ell_k}\left|[i\theta,\ell_{k-1}\theta,\dots,\ell_1\theta+x^*]-[i\theta,\ell_{k-1}\theta,\dots,\ell_1\theta+x]\right|\\
&= \sum_{i\ge\ell_k}\overline{\lambda}_\theta\left\{[i\theta,\ell_{k-1}\theta,\dots,\ell_1\theta+x],[i\theta,\ell_{k-1}\theta,\dots,\ell_1\theta+x^*]\times[0,\theta]\right\}\\
&\le (1+\theta^2)\log\left(1+\theta^2\right)\sum_{i\ge\ell_k}\overline{\gamma}_\theta\left\{[i\theta,\ell_{k-1}\theta,\dots,\ell_1\theta+x],[i\theta,\ell_{k-1}\theta,\dots,\ell_1\theta+x^*]\times[0,\theta]\right\}
\end{aligned}$$

$$= (1+\theta^2)\log\left(1+\theta^2\right)\sum_{i\geq \ell_k}\overline{\gamma}_\theta\left\{(x^*,x)\times I(\ell_1,\ldots,\ell_{k-1})\right\}$$

$$\leq \frac{(1+\theta^2)\log\left(1+\theta^2\right)}{\log\left(1+\theta^2\right)}\sum_{i\geq \ell_k}\overline{\lambda}_\theta\left\{(x^*,x)\times I(\ell_1,\ldots,\ell_{k-1})\right\} \leq \frac{(1+\theta^2)^2}{\theta^2}\overline{\lambda}_\theta(\Delta_{x,y}\setminus\Delta_{x^*,y}). \tag{3.13}$$

Since $\overline{F}_n(x,\theta)=F_n(x)$, from Theorem 3.1 it follows that

$$\begin{aligned}
&\sum_{i\geq \ell_1}\left\{\overline{F}_{n-1}\left(\frac{1}{i\theta+x^*},\theta\right)-\overline{F}_{n-1}\left(\frac{1}{i\theta+x},\theta\right)\right\}\\
&=\sum_{i\geq \ell_1}\left\{F_{n-1}\left(\frac{1}{i\theta+x^*}\right)-F_{n-1}\left(\frac{1}{i\theta+x}\right)\right\}\\
&=\sum_{i\geq \ell_1}\left\{\gamma_\theta\left(\left[\frac{1}{i\theta+x},\frac{1}{i\theta+x^*}\right]\right)+\lambda_\theta\left(\left[\frac{1}{i\theta+x},\frac{1}{i\theta+x^*}\right]\right)\mathcal{O}\left(q^{n-1}\right)\right\}\\
&=\frac{1}{\log\left(1+\theta^2\right)}\log\frac{\ell_1\theta+x}{\ell_1\theta+x^*}+\frac{(1+\theta^2)^2}{\theta^2}\overline{\lambda}_\theta\left(\Delta_{x,y}\setminus\Delta_{x^*,y}\right)\mathcal{O}\left(q^{n-1}\right).
\end{aligned} \tag{3.14}$$

Now from (3.10), (3.14), we have:

$$\begin{aligned}
\overline{F}_n(x,y)-\overline{F}_n(x^*,y)&=\frac{1}{\log\left(1+\theta^2\right)}\log\frac{\ell_1\theta+x}{\ell_1\theta+x^*}+\frac{(1+\theta^2)^2}{\theta^2}\overline{\lambda}_\theta\left(\Delta_{x,y}\setminus\Delta_{x^*,y}\right)\mathcal{O}\left(q^{n-1}\right)\\
&+\left\{\overline{F}_{n-1}\left(\frac{1}{\ell_1\theta+x},y_1\right)-\overline{F}_{n-1}\left(\frac{1}{\ell_1\theta+x^*},y_1\right)\right\}.
\end{aligned}$$

Applying (3.10) and Theorem 3.1 again, we have:

$$\begin{aligned}
&\overline{F}_{n-1}\left(\frac{1}{\ell_1\theta+x},y_1\right)-\overline{F}_{n-1}\left(\frac{1}{\ell_1\theta+x^*},y_1\right)\\
&=\sum_{i\geq \ell_2}\left\{\overline{F}_{n-2}\left(\frac{1}{i\theta+\frac{1}{\ell_1\theta+x^*}},\theta\right)-\overline{F}_{n-2}\left(\frac{1}{i\theta+\frac{1}{\ell_1\theta+x}},\theta\right)\right\}\\
&+\overline{F}_{n-2}\left(\frac{1}{\ell_2\theta+\frac{1}{\ell_1\theta+x^*}},y_2\right)-\overline{F}_{n-2}\left(\frac{1}{\ell_2\theta+\frac{1}{\ell_1\theta+x}},y_2\right)\\
&=\frac{1}{\log\left(1+\theta^2\right)}\log\frac{\ell_2\theta+[\ell_1\theta+x]}{\ell_2\theta+[\ell_1\theta+x^*]}+\frac{(1+\theta^2)^2}{\theta^2}\overline{\lambda}_\theta\left(\Delta_{x,y}\setminus\Delta_{x^*,y}\right)\mathcal{O}\left(q^{n-2}\right)\\
&+\overline{F}_{n-2}\left(\frac{1}{\ell_2\theta+\frac{1}{\ell_1\theta+x^*}},y_2\right)-\overline{F}_{n-2}\left(\frac{1}{\ell_2\theta+\frac{1}{\ell_1\theta+x}},y_2\right).
\end{aligned}$$

Applying (3.10) and Theorem 3.1 d-times and tacking into account that $y_d=0$, we get

$$\begin{aligned}
&\overline{F}_n(x,y)-\overline{F}_n(x^*,y)\\
&=\frac{1}{\log(1+\theta^2)}\times\log\left(\frac{\ell_1\theta+x}{\ell_1\theta+x^*}\cdot\frac{[\ell_1\theta+x]+\ell_2\theta}{[\ell_1\theta+x^*]+\ell_2\theta}\cdot\frac{[\ell_{d-1}\theta,\ldots,\ell_2\theta,\ell_1\theta+x]+\ell_d\theta}{[\ell_{d-1}\theta,\ldots,\ell_2\theta,\ell_1\theta+x^*]+\ell_d\theta}\right)\\
&\quad+\frac{(1+\theta^2)^2}{\theta^2}\overline{\lambda}_\theta\left(\Delta_{x,y}\setminus\Delta_{x^*,y}\right)\left(\mathcal{O}\left(q^{n-1}\right)+\ldots+\mathcal{O}\left(q^{n-d}\right)\right).
\end{aligned}\tag{3.15}$$

Here the constant of the big-$\mathcal{O}$ symbol is C as given in Theorem 3.1.

If p_d and q_d are as in (2.5) and (2.6) with $a_1=\ell_1+x/\theta$ and $a_i=\ell_i$, $i=2,\ldots,d$, then $\ell_1\theta+x=q_1/q_0$, $\ell_2\theta+[\ell_1\theta+x]=q_2/q_1$, $\ell_d\theta+[\ell_{d-1}\theta,\ldots,\ell_2\theta,\ell_1\theta+x]=q_d/q_{d-1}$.

Let p_d^* and q_d^* are as in (2.5) and (2.6), with $a_1=\ell_1+x^*/\theta$ and $a_i=\ell_i$, $i=2,\ldots,d$. Note that $p_d=p_d^*$. Thus we have

$$\begin{aligned}
&\frac{\ell_1\theta+x}{\ell_1\theta+x^*}\cdot\frac{[\ell_1\theta+x]+\ell_2\theta}{[\ell_1\theta+x^*]+\ell_2\theta}\cdot\frac{[\ell_{d-1}\theta,\ldots,\ell_2\theta,\ell_1\theta+x]+\ell_d\theta}{[\ell_{d-1}\theta,\ldots,\ell_2\theta,\ell_1\theta+x^*]+\ell_d\theta}=\frac{q_d}{q_d^*}=\frac{p_d^*}{q_d^*}\frac{q_d}{p_d}\\
&=\frac{x+\ell_1\theta+[\ell_2\theta,\ldots,\ell_d\theta]}{x^*+\ell_1\theta+[\ell_2\theta,\ldots,\ell_d\theta]}=\frac{x+\frac{1}{y}}{x^*+\frac{1}{y}}=\frac{1+xy}{1+x^*y}.
\end{aligned}\tag{3.16}$$

Since $q^{n-d}+\ldots+q^{n-1}=q^{n-d}\dfrac{1-q^d}{1-q}<\dfrac{q^{n-d}}{1-q}$, from (3.15) and (3.16) we obtain the desired result. □

3.3. Proof of Theorem 3.2

Let $(x,y)\in[0,\theta]^2$, $n\geq 2$. In view of Lemma 3.3 we assume that $y\notin\mathbb{Q}$. Put $d:=\max\{\kappa\in\mathbb{N}:\kappa \text{ is even and } \kappa\leq\lfloor n/2\rfloor+2\}$. Using (2.5), (2.6) and (2.7) we obtain

$$\left|y-\frac{p_d}{q_d}\right|\leq\frac{1}{q_dq_{d+1}}<\frac{1}{q_d^2}.\tag{3.17}$$

Furthermore,

$$q_n\geq\overline{q}_n,\quad n\in\mathbb{N}_+,\tag{3.18}$$

where $\overline{q}_n$ satisfies the recurrence relation

$$\overline{q}_n=m\theta\overline{q}_{n-1}+\overline{q}_{n-2},\quad n\in\mathbb{N}_+,\tag{3.19}$$

with $\overline{q}_{-1}=0$ and $\overline{q}_0=1$. It is easy to see that

$$\overline{q}_n=\frac{\theta}{\sqrt{1+4\theta^2}}\left[\left(\frac{1+\sqrt{1+4\theta^2}}{2\theta}\right)^{n+1}-\left(\frac{1-\sqrt{1+4\theta^2}}{2\theta}\right)^{n+1}\right].\tag{3.20}$$

From (3.17), (3.18) and (3.20) and the fact that d is even, we get

$$\left|y-\frac{p_d}{q_d}\right|<\frac{1}{\left(\overline{q}_d\right)^2}<\frac{1+4\theta^2}{\theta^2}\frac{1}{\left(\frac{1+\sqrt{1+4\theta^2}}{2\theta}\right)^n}. \tag{3.21}$$

Since $\Delta_{x,p_d/q_d}\subset\Delta_{x,y}$ and $\overline{F}_n(x,y)=\overline{\lambda}_\theta\left(\left(\overline{T}_\theta\right)^{-n}(\Delta_{x,y})\right)$, from (3.11), (3.21) and the fact that $\overline{T}_\theta$ is $\overline{\gamma}_\theta$-invariant, we find that

$$\begin{aligned}\overline{F}_n(x,y)-\overline{F}_n\left(x,\frac{p_d}{q_d}\right)&=\overline{\lambda}_\theta\left(\left(\overline{T}_\theta\right)^{-n}(\Delta_{x,y})\setminus\left(\overline{T}_\theta\right)^{-n}(\Delta_{x,p_d/q_d})\right)\\&\le\left(1+\theta^2\right)\log\left(1+\theta^2\right)\overline{\gamma}_\theta\left(\left(\overline{T}_\theta\right)^{-n}(\Delta_{x,y})\setminus\left(\overline{T}_\theta\right)^{-n}(\Delta_{x,p_d/q_d})\right)\\&=\left(1+\theta^2\right)\log\left(1+\theta^2\right)\overline{\gamma}_\theta\left(\left(\overline{T}_\theta\right)^{-n}\left(\Delta_{x,y}\setminus\Delta_{x,p_d/q_d}\right)\right)\\&\le\left(1+\theta^2\right)\log\left(1+\theta^2\right)\frac{1}{\log\left(1+\theta^2\right)}\overline{\lambda}_\theta\left([0,x]\times\left[\frac{p_d}{q_d},y\right]\right)\\&=\left(1+\theta^2\right)\frac{x}{\theta}\frac{1}{\theta}\left|y-\frac{p_d}{q_d}\right|<\frac{1+\theta^2}{\theta^2}\frac{1+4\theta^2}{\theta^2}\frac{x}{\left(\frac{1+\sqrt{1+4\theta^2}}{2\theta}\right)^n}.\end{aligned} \tag{3.22}$$

Since for every fixed $x\in[0,\theta]$ the function $y\mapsto\log(1+xy)$ is a differentiable on $[0,\theta]$, by the *Mean Value Theorem* we have

$$\left|\log(1+xy)-\log\left(1+x\frac{p_d}{q_d}\right)\right|=\left|y-\frac{p_d}{q_d}\right|\cdot\left|\frac{x}{1+x\xi}\right|<\frac{1+4\theta^2}{\theta^2}\frac{x}{\left(\frac{1+\sqrt{1+4\theta^2}}{2\theta}\right)^n} \tag{3.23}$$

where $p_d/q_d\le\xi\le y$.

Finally, from Lemma 3.3, (3.22) and (3.23), and since $\overline{F}_n(0,p_d/q_d)=0$, we have:

$$\begin{aligned}\left|\overline{F}_n(x,y)-\frac{\log(1+xy)}{\log\left(1+\theta^2\right)}\right|&\le\left|\overline{F}_n(x,y)-\overline{F}_n\left(x,\frac{p_d}{q_d}\right)\right|\\&+\left|\overline{F}_n\left(x,\frac{p_d}{q_d}\right)-\overline{F}_n\left(0,\frac{p_d}{q_d}\right)-\frac{1}{\log\left(1+\theta^2\right)}\log\left(1+x\frac{p_d}{q_d}\right)\right|\\&+\frac{1}{\log\left(1+\theta^2\right)}\left|\log(1+xy)-\log\left(1+x\frac{p_d}{q_d}\right)\right|\\&\le\frac{1+\theta^2}{\theta^2}\frac{1+4\theta^2}{\theta^2}\frac{x}{\left(\frac{1+\sqrt{1+4\theta^2}}{2\theta}\right)^n}+\frac{\left(1+\theta^2\right)^2}{\theta^2}\frac{C}{1-q}\frac{x}{\theta^2}\frac{p_d}{q_d}q^{n-d}\\&+\frac{1}{\log\left(1+\theta^2\right)}\frac{1+4\theta^2}{\theta^2}\frac{x}{\left(\frac{1+\sqrt{1+4\theta^2}}{2\theta}\right)^n}\end{aligned}$$

$$\leq \frac{1+4\theta^2}{\theta}\left(\frac{1+\theta^2}{\theta^2}+\frac{1}{\log(1+\theta^2)}\right)\frac{1}{\left(\frac{1+\sqrt{1+4\theta^2}}{2\theta}\right)^n}$$
$$+\frac{\left(1+\theta^2\right)^2}{\theta^2}\frac{C}{1-q}q^{n-d}.$$

Since $d \leq n/2+2$, we have

$$\left|\overline{F}_n(x,y)-\frac{\log(1+xy)}{\log(1+\theta^2)}\right| \leq \frac{1+4\theta^2}{\theta}\left(\frac{1+\theta^2}{\theta^2}+\frac{1}{\log(1+\theta^2)}\right)\frac{1}{\left(\frac{1+\sqrt{1+4\theta^2}}{2\theta}\right)^n}$$
$$+\frac{\left(1+\theta^2\right)^2}{\theta^2}\frac{C(\sqrt{q})^n}{(1-q)q^2} \leq \mathcal{K}\rho^n$$

where

$$\mathcal{K}=\frac{1+4\theta^2}{\theta}\left(\frac{1+\theta^2}{\theta^2}+\frac{1}{\log(1+\theta^2)}\right)+\frac{\left(1+\theta^2\right)^2}{\theta^2}\frac{C}{(1-q)q^2}$$

and

$$\rho=\max\left(\sqrt{q},\frac{2\theta}{1+\sqrt{1+4\theta^2}}\right)<1$$

and the proof is complete. □

4. A two-dimensional Gauss–Kuzmin theorem

In this section we shall estimate the error term

$$e_{n,a}(x,y)=\gamma_{\theta,a}\left(T_\theta^n\in[0,x],s_{n,a}\in[0,y]\right)-\frac{\log(1+xy)}{\log(1+\theta^2)}$$

for any $a\in[0,\theta]$, $x,y\in[0,\theta]$ and $n\in\mathbb{N}_+$.

In the main result of this section, Theorem 4.4, we shall derive lower and upper bounds (not depending on $a\in[0,\theta]$) of the supremum

$$\sup_{x,y\in[0,\theta]}|e_{n,a}(x,y)|,\quad a\in[0,\theta], \tag{4.1}$$

which provide a more refined estimate of the convergence rate involved. First, we obtain a lower bound for the following approximation error.

Theorem 4.1. *For any $a\in[0,\theta]$ and $n\in\mathbb{N}_+$ we have*

$$\frac{1}{2}P_{m(n)}(\theta)\leq\sup_{y\in[0,\theta]}\left|\gamma_{\theta,a}\left(s_{n,a}\leq y\right)-\gamma_\theta\left([0,y]\right)\right|$$

with $P_{m(n)}(\theta) = P_{i_1 \dots i_n}(\theta)$ *as in (2.36), where we write* $m(n)$ *for* $(i_1, \dots, i_n)$ *with* $i_1 = \dots = i_n = m$, $m \in \mathbb{N}_+$.

Proof. The continuity of the function $y \to \gamma_\theta([0,y])$, $y \in [0,\theta]$ and the equation

$$\lim_{h \searrow 0} \gamma_{\theta,a}(s_{n,a} \le y - h) = \gamma_{\theta,a}(s_{n,a} \le y)$$

imply that

$$\sup_{y \in [0,\theta]} |\gamma_{\theta,a}(s_{n,a} \le y) - \gamma_\theta([0,y])| = \sup_{y \in [0,\theta]} |\gamma_{\theta,a}(s_{n,a} < y) - \gamma_\theta([0,y])|$$

for any $a \in [0,\theta]$ and $n \in \mathbb{N}_+$. For any $s \in [0,\theta]$ we then have

$$\begin{aligned}\gamma_{\theta,a}(s_{n,a} = s) &= \gamma_{\theta,a}(s_{n,a} \le s) - \gamma_\theta([0,s]) - (\gamma_{\theta,a}(s_{n,a} < s) - \gamma_\theta([0,s])) \\ &\le \sup_{y \in [0,\theta]} |\gamma_{\theta,a}(s_{n,a} \le y) - \gamma_\theta([0,y])| + \sup_{y \in [0,\theta]} |\gamma_{\theta,a}(s_{n,a} < y) - \gamma_\theta([0,y])| \\ &= 2 \sup_{y \in [0,\theta]} |\gamma_{\theta,a}(s_{n,a} \le y) - \gamma_\theta([0,y])|.\end{aligned}$$

Hence

$$\sup_{y \in [0,\theta]} |\gamma_{\theta,a}(s_{n,a} \le y) - \gamma_\theta([0,y])| \ge \frac{1}{2} \sup_{s \in [0,\theta]} \gamma_{\theta,a}(s_{n,a} = s),$$

for any $a \in [0,\theta]$ and $n \in \mathbb{N}_+$. Next, using (2.36) we have

$$\gamma_{\theta,a}(s_{n,a} = [i_n\theta, \dots, i_2\theta, i_1\theta + a]) = P_{i_1 \dots i_n}(a), \ n \ge 2,$$
$$\gamma_{\theta,a}\left(s_{1,a} = \frac{1}{i_1\theta + a}\right) = P_{i_1}(a)$$

for any $a \in [0,\theta]$ and $i_1, \dots, i_n \in \mathbb{N}_m$. By (2.37) we have

$$\sup_{s \in [0,\theta]} \gamma_{\theta,a}(s_{n,a} = s) = P_{m(n)}(a), \quad a \in [0,\theta].$$

By the same equation we have

$$P_{m(n)}(a) = \frac{1 + a\theta}{q_{n-1}(\underbrace{m, \dots, m}_{(n-1)\ times})(a + m\theta) + p_{n-1}(\underbrace{m, \dots, m}_{(n-1)\ times})}$$
$$\times \frac{1}{q_n(\underbrace{m, \dots, m, m}_{n\ times})(a + m\theta) + p_n(\underbrace{m, \dots, m, m}_{n\ times})}.$$

It is easy to see that $P_{m(n)}(\cdot)$ is a decreasing function. Therefore

$$\sup_{s\in[0,\theta]} \gamma_{\theta,a}\left(s_{n,a}=s\right) \geq P_{m(n)}(\theta), \quad n\in\mathbb{N}_+$$

for any $a\in[0,\theta]$. □

Theorem 4.2 *(The lower bound). For any $a\in[0,\theta]$ and $n\in\mathbb{N}_+$ we have*

$$\frac{1}{2}P_{m(n)}(\theta) \leq \sup_{x,y\in[0,\theta]} \left|\gamma_{\theta,a}\left(T_\theta^n\in[0,x], s_{n,a}\in[0,y]\right) - \frac{\log(1+xy)}{\log\left(1+\theta^2\right)}\right|.$$

Proof. For any $a\in[0,\theta]$ and $n\in\mathbb{N}_+$, by Theorem 4.1 we have

$$\begin{aligned}
&\sup_{x,y\in[0,\theta]} \left|\gamma_{\theta,a}\left(T_\theta^n\in[0,x], s_{n,a}\in[0,y]\right) - \frac{\log(1+xy)}{\log\left(1+\theta^2\right)}\right| \\
&\geq \sup_{y\in[0,\theta]} \left|\gamma_{\theta,a}\left(T_\theta^n\in[0,\theta], s_{n,a}\in[0,y]\right) - \frac{\log(1+\theta y)}{\log\left(1+\theta^2\right)}\right| \\
&= \sup_{y\in[0,\theta]} \left|\gamma_{\theta,a}\left(s_{n,a}\in[0,y]\right) - \gamma_\theta\left([0,y]\right)\right| \geq \frac{1}{2}P_{m(n)}(\theta). \quad \square
\end{aligned}$$

In what follows we use the characteristic properties of the transition operator associated with the RSCC underlying θ-expansions. By restricting this operator to the Banach space of functions of bounded variation on $[0,\theta]$, we derive an explicit upper bound for the supremum (4.1).

Theorem 4.3 *(The upper bound). For any $a\in[0,\theta]$ and $n\in\mathbb{N}$ we have*

$$\sup_{x,y\in[0,\theta]} \left|\gamma_{\theta,a}\left(T_\theta^n\in[0,x], s_{n,a}\in[0,y]\right) - \frac{\log(1+xy)}{\log\left(1+\theta^2\right)}\right| \leq \frac{1}{(m+1)^n}.$$

Proof. Let $F_{n,a}(y)=\gamma_{\theta,a}(s_{n,a}\leq y)$ and $G_{n,a}(y)=F_{n,a}(y)-\gamma_\theta([0,y])$, $a,y\in[0,\theta]$, $n\in\mathbb{N}$. As we have noted U is the transition operator of the Markov chain $(s_{n,a})_{n\in\mathbb{N}}$ on $\left([0,\theta],\mathcal{B}_{[0,\theta]},\gamma_{\theta,a}\right)$ for any $a\in[0,\theta]$. For any $y\in[0,\theta]$ consider the function f_y defined on $[0,\theta]$ as

$$f_y(a) := \begin{cases} 1 & \text{if } 0\leq a\leq y, \\ 0 & \text{if } y<a\leq\theta. \end{cases}$$

Hence

$$U^n f_y(a) = E_a\left(f_y(s_{n,a})\middle|\, s_{0,a}=a\right) = \gamma_{\theta,a}(s_{n,a}\leq y)$$

for all $a,y\in[0,\theta]$, $n\in\mathbb{N}$. As

$$U_\infty f_y = \int_0^\theta f_y(a)\gamma_\theta(\mathrm{da}) = \gamma_\theta([0,y]), \quad y \in [0,\theta].$$

It follows from Proposition 2.2 that

$$\begin{aligned} |G_{n,a}(y)| &= |\gamma_{\theta,a}(s_{n,a} \le y) - \gamma_\theta([0,y])| = |U^n f_y(a) - U_\infty f_y| \\ &\le \frac{1}{(m+1)^n} \operatorname{var} f_y = \frac{1}{(m+1)^n} \end{aligned} \tag{4.2}$$

for all $a, y \in [0,\theta]$, $n \in \mathbb{N}$. By (2.26), for all $a, x, y \in [0,\theta]$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} \gamma_{\theta,a}\left(T_\theta^n \in [0,x], s_{n,a} \in [0,y]\right) &= \int_0^y \gamma_{\theta,a}\left(T_\theta^n \in [0,x] | \, s_{n,a} = z\right) \mathrm{dF_{n,a}(z)} \\ &= \int_0^y \frac{(z\theta+1)x}{(zx+1)\theta} \mathrm{dF_{n,a}(z)} \\ &= \int_0^y \frac{(z\theta+1)x}{(zx+1)\theta} \gamma_\theta(\mathrm{dz}) + \int_0^y \frac{(z\theta+1)x}{(zx+1)\theta} \mathrm{dG_{n,a}(z)} \\ &= \frac{\log(1+xy)}{\log(1+\theta^2)} + \frac{(z\theta+1)x}{(zx+1)\theta} \, G_{n,a}(z)\Big|_0^y \\ &\quad - \int_0^y \frac{x(\theta-x)}{(zx+1)^2\theta} G_{n,a}(z)\mathrm{dz}. \end{aligned}$$

Hence, by (4.2)

$$\begin{aligned} \left|\gamma_{\theta,a}\left(T_\theta^n \in [0,x], s_{n,a} \in [0,y]\right) - \frac{\log(1+xy)}{\log(1+\theta^2)}\right| &\le \frac{1}{(m+1)^n}\left(\frac{(y\theta+1)x}{(xy+1)\theta} - \frac{(\theta-x)xy}{(xy+1)\theta}\right) \\ &= \frac{x}{(m+1)^n\theta} \le \frac{1}{(m+1)^n} \end{aligned}$$

for all $a, x, y \in [0,\theta]$ and $n \in \mathbb{N}$. □

Combining Theorem 4.2 with Theorem 4.3 we obtain Theorem 4.4.

Theorem 4.4. *For any $a \in [0,\theta]$ and $n \in \mathbb{N}_+$ we have*

$$\frac{1}{2} P_{m(n)}(\theta) \le \sup_{x,y\in[0,\theta]} \left|\gamma_{\theta,a}\left(T_\theta^n \in [0,x], s_{n,a} \in [0,y]\right) - \frac{\log(1+xy)}{\log(1+\theta^2)}\right| \le \frac{1}{(m+1)^n}.$$

5. Final remarks

To conclude this paper, we note that

$$P_{m(n)}(\theta) = \frac{m+1}{\overline{q}_{n+1}\overline{q}_{n+2}}, \quad n \in \mathbb{N}_+,$$

where $\overline{q}_n$ is defined in (3.19).

Using (3.20), it should be noted that Theorem 4.4 in connection with the limits

$$\lim \left(\frac{1}{2}P_{m(n)}(\theta)\right)^{1/n} = \frac{2\theta^2}{1+2\theta^2+\sqrt{1+4\theta^2}},$$
$$\lim \left(\frac{1}{(m+1)^n}\right)^{1/n} = \frac{1}{m+1},$$

leads to an estimate of the order of magnitude of the supremum (4.1). Actually, Theorem 4.4 implies that the convergence rate is $\mathcal{O}(\alpha^n)$, with

$$\frac{2\theta^2}{1+2\theta^2+\sqrt{1+4\theta^2}} \leq \alpha \leq \frac{1}{m+1}.$$

For example, we have

$m = 1$	$0.381966 \leq \alpha \leq 0.50000$
$m = 2$	$0.267949192 \leq \alpha \leq 0.33333\ldots$
$m = 3$	$0.208711948 \leq \alpha \leq 0.25000\ldots$
$m = 10$	$0.083920216 \leq \alpha \leq 0.090909\ldots$
$m = 100$	$0.009804864 \leq \alpha \leq 0.00990099$
$m = 1000$	$0.000998004 \leq \alpha \leq 0.000999$
$m = 10000$	$0.00009998 \leq \alpha \leq 0.00009999$

Also, Fig. 2 suggests that for very large values of m the lower and upper bounds are very close.

The red graph and the blue graph represent the lower and upper bound graphs, where

$$f_{red} : \mathbb{N}_+ \to \mathbb{R}, \ f_{red}(m) = \frac{2}{m+2+\sqrt{m(m+4)}}$$

and, respectively

$$f_{blue} : \mathbb{N}_+ \to \mathbb{R}, \ f_{blue}(m) = \frac{1}{m+1}.$$

Acknowledgment

The authors would like to thank the referee for carefully reading our manuscript and for giving such constructive comments which substantially helped improving the quality of the paper.

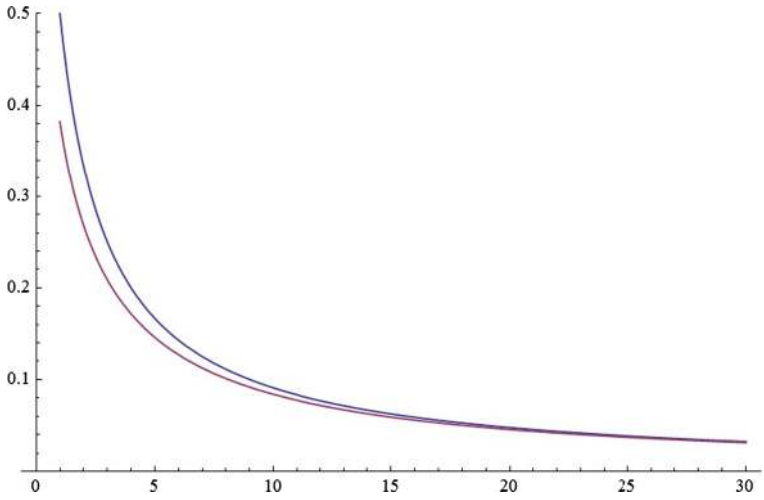

Fig. 2. Graphs of lower and upper bounds. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

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ACTA ARITHMETICA
193.3 (2020)

A Gauss–Kuz'min–Lévy theorem for Rényi-type continued fractions

by

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1. Introduction. The present paper continues a series of papers dedicated to Rényi-type continued fraction expansions [LS19, SL19]. Actually, these continued fractions are a particular case of *u-backward continued fractions* studied by Gröchenig and Haas [GH96]. In 1957, Rényi [R57] showed that every irrational number $x \in [0,1)$ has an infinite continued fraction expansion of the form

$$x = 1 - \cfrac{1}{n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 - \ddots}}} =: [n_1, n_2, n_3, \ldots]_b, \tag{1.1}$$

where each n_i is an integer greater than 1. We call the expansion in (1.1) a *backward continued fraction*. The underlying dynamical system is the Rényi map R defined from $[0,1)$ to $[0,1)$ by

$$R(x) := \frac{1}{1-x} - \left\lfloor \frac{1}{1-x} \right\rfloor, \tag{1.2}$$

which has a neutral fixed point at 0 and thus is nonuniformly hyperbolic. Rényi showed that the infinite measure $\mathrm{d}x/x$ is invariant for R. This map does not possess a finite absolutely continuous invariant measure, and the usual inducing trick to study its thermodynamic formalism does not work.

Unlike (1.2), the Gauss map defined from $(0,1]$ to $(0,1]$ by

$$G(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \tag{1.3}$$

2020 *Mathematics Subject Classification*: Primary 11J70; Secondary 60A10.
Key words and phrases: Rényi continued fractions, Gauss–Kuz'min–Lévy problem, Hurwitz zeta function.
Received 9 October 2018; revised 7 February 2019 and 4 December 2019.
Published online 27 January 2020.

DOI: 10.4064/aa181009-18-2

which generates the well-known *regular continued fraction expansion*, does have a finite absolutely continuous invariant measure, namely the *Gauss measure* $\mathrm{d}x/(x+1)$. The Gauss map is uniformly expanding, but has infinitely many branches. The graph of R can be obtained from that of G by reflecting the latter in the line $x = 1/2$ (see Figure 1). It is for this reason that the continued fraction (1.1) has been called "backward".

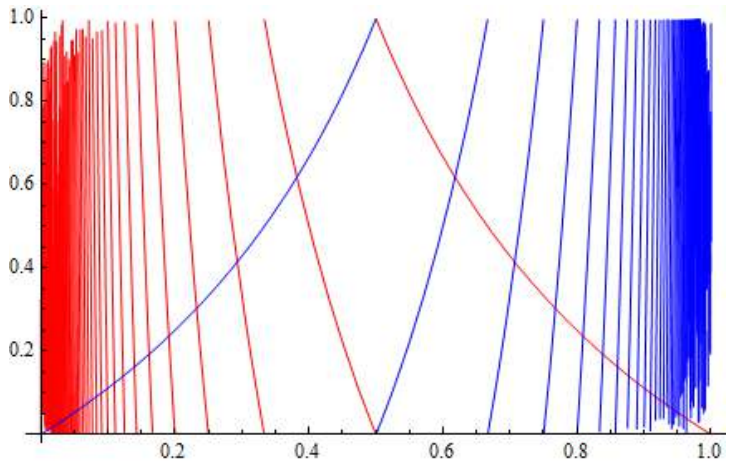

Fig. 1. Graphs of Gauss (piecewise decreasing, red in the pdf file) and Rényi (piecewise increasing, blue) transformations

Starting from the expansion in (1.1) and the Rényi transformation R, Gröchenig and Haas [GH96] define the family of maps $T_u(x) := \frac{1}{u(1-x)} - \left\lfloor \frac{1}{u(1-x)} \right\rfloor$, where $u > 0$, $x \in [0,1)$. Given $u \in (0,4)$ and $x \in [0,1)$, x has the *u-backward continued fraction expansion*

$$x = 1 - \cfrac{1}{un_1 - \cfrac{1}{n_2 - \cfrac{1}{un_3 - \cfrac{1}{n_4 - \ddots}}}} =: [a_1, a_2, a_3, \ldots]_u, \tag{1.4}$$

where the integers $n_i = 1 + a_i$ are ≥ 2 and the coefficient of n_i is 1 or u, depending on the parity of i. In the particular case $u = 1/N$, for a positive integer $N \geq 2$, they have identified a finite absolutely continuous invariant measure for T_u, namely $\mathrm{d}x/(x+N-1)$. For $u = 1/N$, where $N \geq 2$ is an integer, we will call T_u the *Rényi-type continued fraction transformation* and denote it by R_N.

The metrical theory of this algorithm was initiated in [LS19]. The first known metrical problem concerning (regular) continued fractions is due to Gauss. At the start of the 20th century an old discovery of Gauss was shown

to tie the theory of continued fractions to probability theory and ergodic theory. In 1802 and 1812 Gauss found the invariant measure of the transformation G in (1.3) underlying the regular continued fraction, and asked Lagrange in a letter in 1812 how fast $\lambda(G^{-n}([0,x]))$ converges to the invariant measure $\gamma([0,x]) = \log(1+x)/\log 2$. Here λ is the Lebesgue measure. In 1928, Kuz'min [K28] answered Gauss' question by giving an estimate of the remainder. Independently in 1929 Paul Lévy [L29] improved Kuz'min's result and published another proof. In the 60's Szüsz [S61] was able to prove the same result by using Kuz'min's approach.

In [LS19] we started an approach to the metrical theory of Rényi-type continued fraction expansions via dependence with complete connections. More precisely, we obtained a version of the Gauss–Kuz'min theorem for these expansions by applying the theory of random systems with complete connections, due to Iosifescu [IG09]. Once a finite ergodic measure ρ_N is obtained for the map R_N (recall that R_N is ergodic and measure preserving under ρ_N), classical results of ergodic theory, such as the Birkhoff ergodic theorem, yield precise information on the frequency with which a digit occurs. It should be stressed that the ergodic theorem does not yield any information on the convergence rate in the Gauss–Kuz'min problem that amounts to determining the asymptotic behavior of $\mu(R_N^{-n})$ as $n \to \infty$, where μ is an arbitrary probability measure. So that a Gauss–Kuz'min theorem is needed. Using the natural extensions for Rényi-type transformations, we obtained an infinite-order-chain representation $(\overline{a}_\ell)_{\ell\in\mathbb{Z}}$ of the sequence $(a_n)_{n\in\mathbb{N}_+}$ of incomplete quotients of these expansions. Then we showed that the associated random systems with complete connections are with contraction and their transition operators are regular with respect to the Banach space of Lipschitz functions. This allowed us to solve a variant of the Gauss–Kuz'min problem. In this result the constants involved are far from optimal. So the hunt for the best possible constants started.

In [SL19] we used a Wirsing-type approach [W74] to get close to the optimal convergence rate. Let us notice that Wirsing's Gauss–Kuz'min–Lévy theorem leads to a good estimation of the convergence rate to 0 as $n \to \infty$ of the ψ-mixing coefficients of the incomplete quotients $(a_n)_{n\in\mathbb{N}_+}$ under ρ_N or of the extended versions $(\overline{a}_\ell)_{\ell\in\mathbb{Z}}$ under the extended measure $\overline{\rho}_N$ (see [LS19]). Paralleling the treatment in the case of regular continued fractions, the Gauss–Kuz'min–Lévy problem for the transformation R_N can be approached in terms of the Perron–Frobenius operator under the invariant measure induced by the limit distribution function. By restricting the domain of the associated Perron–Frobenius operator of R_N under its invariant measure ρ_N to the Banach space of functions which have a continuous derivative on $[0,1]$, we obtained upper and lower bounds of the error which provide a refined estimate of the convergence rate. For example, in case $N = 100$, the upper and lower bounds of the convergence rate are respec-

tively $\mathcal{O}(w_{100}^n)$ and $\mathcal{O}(v_{100}^n)$ as $n \to \infty$, with $v_{100} > 0.00503350150708559$ and $w_{100} < 0.00503358526129032$.

The purpose of this paper is to continue our investigation on the asymptotic behavior of the distribution functions of Rényi-type transformations. In order to prove a Gauss–Kuz'min–Lévy-type theorem for Rényi-type continued fraction expansions, we apply the method of Szüsz [S61]. We mention that using this method, we obtain more information on the convergence rate involved. The main novelty of this paper is the explicit expression in terms of Hurwitz zeta functions of η_N that appears in Theorem 3.1. In addition, the estimate we have for η_N shows that $\eta_N \to 0$ as $N \to \infty$. Finally, to enable direct comparisons of the results obtained in the last two methods (Wirsing and Szüsz), we give upper and lower bounds of η_N for $N = 100$: $0.00505050495049505 < \eta_{100} < 0.0050753806723955975$.

2. Rényi-type continued fractions. In this section we briefly present known results about Rényi-type continued fractions.

Fix an integer $N \geq 2$. The *Rényi-type continued fraction transformation* $R_N : [0,1] \to [0,1]$ is given by

$$R_N(x) = \frac{N}{1-x} - \left\lfloor \frac{N}{1-x} \right\rfloor, \quad x \neq 1; \qquad R_N(1) = 0. \tag{2.1}$$

For any irrational $x \in [0,1]$, R_N generates a new continued fraction expansion of x of the form

$$x = 1 - \cfrac{N}{1 + a_1 - \cfrac{N}{1 + a_2 - \cfrac{N}{1 + a_3 - \ddots}}} =: [a_1, a_2, a_3, \ldots]_R. \tag{2.2}$$

Here, a_n's are nonnegative integers greater than or equal to N defined by

$$a_1 := a_1(x) = \left\lfloor \frac{N}{1-x} \right\rfloor, \quad x \neq 1; \qquad a_1(1) = \infty, \tag{2.3}$$

and

$$a_n := a_n(x) = a_1(R_N^{n-1}(x)), \qquad n \geq 2, \tag{2.4}$$

with $R_N^0(x) = x$.

The rational approximants to x arise in a manner similar to that in the case of other continued fraction algorithms. In particular we define two integer sequences by $p_0 = 1$, $q_0 = 1$, $p_1 = 1 + a_1 - N$, $q_1 = 1 + a_1$,

$$p_n = (1 + a_n)p_{n-1} - Np_{n-2} \quad \text{and} \quad q_n = (1 + a_n)q_{n-1} - Nq_{n-2}$$

for $n \geq 2$. A simple inductive argument gives

$$p_{n-1}q_n - p_nq_{n-1} = N^n, \qquad n \in \mathbb{N}_+ := \{1, 2, \ldots\}.$$

The rationals p_n/q_n, $n \in \mathbb{N}_+$, are the convergents to x in $[0,1]$. In [GH96] it was shown that the dynamical system $([0,1], \mathcal{B}_{[0,1]}, R_N, \rho_N)$ is measure preserving and ergodic. Here, $\mathcal{B}_{[0,1]}$ denotes the σ-algebra of all Borel subsets of $[0,1]$, and the probability measure ρ_N is defined by

$$\rho_N(A) := \frac{1}{\log\left(\frac{N}{N-1}\right)} \int_A \frac{\mathrm{d}x}{x+N-1}, \qquad A \in \mathcal{B}_{[0,1]}. \tag{2.5}$$

In [LS19] we investigated the Perron–Frobenius operator of R_N on the measurable space $([0,1], \mathcal{B}_{[0,1]}, \mu)$ such that the probability measure μ satisfies $\mu(R_N^{-1}(A)) = 0$ whenever $\mu(A) = 0$ for $A \in \mathcal{B}_{[0,1]}$. In particular, we studied the Perron–Frobenius operator U of $([0,1], \mathcal{B}_{[0,1]}, \rho_N, R_N)$, that is, U is a unique operator on $L^1([0,1], \rho_N) := \{f : [0,1] \to \mathbb{C} : \int_0^1 |f| \,\mathrm{d}\rho_N < \infty\}$ which satisfies

$$\int_A Uf \,\mathrm{d}\rho_N = \int_{R_N^{-1}(A)} f \,\mathrm{d}\rho_N \quad \text{for any } A \in \mathcal{B}_{[0,1]},\ f \in L^1([0,1], \rho_N). \tag{2.6}$$

Also, we have found an explicit formula for the Perron–Frobenius operator under the invariant measure ρ_N, namely

$$Uf(x) = \sum_{i \geq N} P_{N,i}(x) f(u_{N,i}(x)), \qquad f \in L^1([0,1], \rho_N), \tag{2.7}$$

where $P_{N,i}$ and $u_{N,i}$ are functions defined on $[0,1]$ by

$$P_{N,i}(x) := \frac{x+N-1}{(x+i)(x+i-1)}, \tag{2.8}$$

$$u_{N,i}(x) := 1 - \frac{N}{x+i}. \tag{2.9}$$

A more thorough account of Rényi-type continued fractions can be found in [GH96, LS19, SL19].

3. Main result. In this section we show our main theorem. Let μ be a nonatomic probability measure on $\mathcal{B}_{[0,1]}$ and define

$$F_{N,0}(x) := \mu([0,x]), \qquad x \in [0,1], \tag{3.1}$$

$$F_{N,n}(x) := \mu(R_N^n \leq x), \quad x \in [0,1],\ n \in \mathbb{N}_+. \tag{3.2}$$

Main Theorem 3.1 (A Gauss–Kuz'min–Lévy-type theorem). *Let R_N and $F_{N,n}$ be as in* (2.1) *and* (3.2). *Then there exists a constant $0 < \eta_N < 1$ such that F_n can be written as*

$$F_{N,n}(x) = \frac{1}{\log\left(\frac{N}{N-1}\right)} \log\left(\frac{x+N-1}{N-1}\right) + \mathcal{O}(\eta_N^n) \tag{3.3}$$

uniformly with respect to $x \in [0,1]$.

REMARK 3.2. From (3.3), we see that

$$\lim_{n\to\infty} F_{N,n}(x) = \rho_N([0,x]), \tag{3.4}$$

where ρ_N is the measure defined in (2.5). In fact, Theorem 3.1 estimates the error

$$e_{N,n}(x) = e_{N,n}(x,\mu) = \mu(R_N^n \le x) - \rho_N([0,x]), \quad x \in [0,1]. \tag{3.5}$$

To prove Theorem 3.1 we need the following results.

LEMMA 3.3. *For functions $F_{N,n}$ in* (3.2)*, the following Gauss–Kuz'min-type equation holds:*

$$F_{N,n+1}(x) = \sum_{i\ge N}\left\{F_{N,n}\left(1-\frac{N}{x+i}\right) - F_{N,n}\left(1-\frac{N}{i}\right)\right\} \tag{3.6}$$

for $x \in [0,1]$ and $n \in \mathbb{N}$.

Proof. From (2.1) and (2.4), we see that

$$R_N^n(x) = 1 - \frac{N}{a_{n+1} + R_N^{n+1}}, \quad n \in \mathbb{N}_+. \tag{3.7}$$

Now,

$$\begin{aligned} F_{N,n+1}(x) = \mu(R_N^{n+1} \le x) &= \sum_{i\ge N} \mu\left(1-\frac{N}{i} \le R_N^{n+1} \le 1-\frac{N}{i+x}\right) \\ &= \sum_{i\ge N}\left\{F_{N,n}\left(1-\frac{N}{x+i}\right) - F_{N,n}\left(1-\frac{N}{i}\right)\right\}. \ \blacksquare \end{aligned}$$

REMARK 3.4. Assume that for some $p \in \mathbb{N}$, the derivative $F'_{N,p}$ exists everywhere in $[0,1]$ and is bounded. Then it is easy to see by induction that $F'_{N,p+n}$ exists and is bounded for all $n \in \mathbb{N}_+$. This allows us to differentiate (3.6) term by term, obtaining

$$F'_{N,n+1}(x) = \sum_{i\in N} \frac{N}{(x+i)^2} F'_{N,n}\left(1-\frac{N}{x+i}\right). \tag{3.8}$$

We introduce functions $f_{N,n}$ as follows:

$$f_{N,n}(x) := (x+N-1)F'_{N,n}(x), \quad x \in [0,1],\ n \in \mathbb{N}. \tag{3.9}$$

Then (3.8) is

$$f_{N,n+1}(x) = \sum_{i\ge N} P_{N,i}(x) f_{N,n}(u_{N,i}(x)), \tag{3.10}$$

where $P_{N,i}(x)$ and $u_{N,i}(x)$ are given in (2.8) and (2.9), respectively.

LEMMA 3.5. *For* $\{f_{N,n}\}$ *in* (3.9), *define* $M_{N,n} := \max_{x\in[0,1]} |f'_{N,n}(x)|$. *Then*

$$M_{N,n+1} \le \eta_N \cdot M_{N,n} \tag{3.11}$$

where

$$\eta_N = \sum_{i\ge N}\left(\frac{1}{i^3} + \frac{N}{i^2(i+1)}\right). \tag{3.12}$$

Proof. Since

$$P_{N,i}(x) = \frac{i+1-N}{x+i} - \frac{i-N}{x+i-1},$$

we have

$$\begin{aligned} f'_{N,n+1}(x) &= \sum_{i\ge N}\{P'_{N,i}(x) f_{N,n}(u_{N,i}(x)) \\ &\qquad + P_{N,i}(x) f'_{N,n}(u_{N,i}(x)) u'_{N,i}(x)\} \\ &= \sum_{i\ge N}\left\{\left(\frac{i-N}{(x+i-1)^2} - \frac{i+1-N}{(x+i)^2}\right) f_{N,n}(u_{N,i}(x)) \right. \\ &\qquad \left. + P_{N,i}(x) f'_{N,n}(u_{N,i}(x)) \frac{N}{(x+i)^2}\right\} \\ &= \sum_{i\ge N}\left\{\frac{i+1-N}{(x+i)^2}[f_{N,n}(u_{N,i+1}(x)) - f_{N,n}(u_{N,i}(x))] \right. \\ &\qquad \left. + P_{N,i}(x) f'_{N,n}(u_{N,i}(x)) \frac{N}{(x+i)^2}\right\} \\ &= \sum_{i\ge N}\left\{\frac{i+1-N}{(x+i)^3(x+i+1)} f'_{N,n}(\theta_i) \right. \\ &\qquad \left. + f'_{N,n}(u_{N,i}(x)) \frac{N P_{N,i}(x)}{(x+i)^2}\right\} \end{aligned} \tag{3.13}$$

where $u_{N,i+1}(x) < \theta_i < u_{N,i}(x)$. Now (3.13) implies

$$\begin{aligned} M_{N,n+1} \le M_{N,n} \cdot \max_{x\in[0,1]}\Bigg(&\sum_{i\ge N}\frac{i+1-N}{(x+i)^3(x+i+1)} \\ &+ N\sum_{i\ge N}\frac{x+N-1}{(x+i)^3(x+i-1)}\Bigg). \end{aligned} \tag{3.14}$$

We must now calculate the maximum value of the sums in this expression. Since $x \in [0,1]$ and $i \ge N$, we get

$$\frac{i+1-N}{(x+i)^3(x+i+1)} \le \frac{i+1-N}{i^3(i+1)}$$

and

$$\frac{x+N-1}{(x+i)^3(x+i-1)} \le \frac{1}{(x+i)^3} \le \frac{1}{i^3}.$$

Thus,

$$M_{N,n+1} \leq M_{N,n} \cdot \sum_{i \geq N} \left(\frac{1}{i^3} + \frac{N}{i^2(i+1)} \right) \tag{3.15}$$

and the proof is complete. ∎

Proof of Theorem 3.1. Let $\{F_{N,n}\}$ and $\{f_{N,n}\}$ be as in (3.2) and (3.9), respectively. Then

$$F'_{N,n}(x) = \frac{1}{x+N-1} f_{N,n}(x), \quad x \in [0,1],\ n \in \mathbb{N}. \tag{3.16}$$

If we can show the existence of a constant $0 < \eta_N < 1$ such that

$$f_{N,n}(x) = \frac{1}{\log\left(\frac{N}{N-1}\right)} + \mathcal{O}(\eta_N^n), \tag{3.17}$$

then integrating (3.16) we will establish (3.3). To demonstrate that $f_{N,n}(x)$ has the desired form, it suffices to establish that $f'_{N,n}(x) = \mathcal{O}(\eta_N^n)$, as the $1/\log\left(\frac{N}{N-1}\right)$ constant in (3.17) will follow from the normalization requirement that $F_{N,n}(0) = 0$ and $F_{N,n}(1) = 1$.

It remains to prove the following lemma.

LEMMA 3.6. *For every integer $N \geq 2$ there exists a constant $0 < \eta_N < 1$ such that*

$$f'_{N,n}(x) = \mathcal{O}(\eta_N^n), \quad x \in [0,1],\ n \in \mathbb{N}. \tag{3.18}$$

Moreover, for any integer $N \geq 2$,

$$\frac{1}{N^3} + \frac{1}{2N(N+1)} + \frac{1}{2N} < \eta_N < \frac{1}{2N(N-1)} + \frac{1}{N} - \frac{1}{2N+1}. \tag{3.19}$$

Proof. Let η_N be as in Lemma 3.5. Using that lemma, to show (3.18) it is enough to prove that $\eta_N < 1$. First, we will write η_N in terms of Hurwitz zeta functions:

$$\eta_N = \sum_{i \geq N} \left(\frac{1}{i^3} + \frac{N}{i^2(i+1)} \right) = \sum_{i \geq N} \left(\frac{1}{i^3} + \frac{N}{i^2} \right) - 1 = \zeta(3,N) + N\zeta(2,N) - 1.$$

For $i \geq N$ and $a := \frac{1}{2}(\sqrt{4N^2+1} - (2N+1)) > -\frac{1}{2}$, we have

$$a^2 + (2N+1)a + N = 0$$

and

$$a^2 + (2N+1)a + N + (2a+1)(i-N) \geq 0,$$

i.e.,

$$i^2 \leq (i+a)(i+1+a).$$

Hence,

$$\zeta(2,N) \geq \sum_{i\geq N}\left(\frac{1}{i+a}-\frac{1}{i+1+a}\right)=\frac{1}{N+a}=\frac{2}{\sqrt{4N^2+1}-1}.$$

Also we have

$$\zeta(2,N) < \frac{1}{N^2}+\sum_{i\geq N+1}\frac{1}{(i-1/2)(i+1/2)}=\frac{1}{N^2}+\frac{2}{2N+1}.$$

For $i \geq N$ and $b := N(\sqrt{N^2+1}-N) < \frac{1}{2}$, we have

$$b^2+2N^2b-N^2=0$$

and

$$b^2+2N^2b-N^2+(2b-1)(i^2-N^2)\leq 0,$$

i.e.,

$$i^4 \geq (i^2-i+b)(i^2+i+b).$$

Hence

$$\begin{aligned}\zeta(3,N) &< \frac{1}{2}\sum_{i\geq N}\left(\frac{1}{(i-1)i+b}-\frac{1}{i(i+1)+b}\right)=\frac{1}{2(N^2-N+b)}\\ &= \frac{1}{2N(\sqrt{N^2+1}-1)}.\end{aligned}$$

Also, we have

$$\begin{aligned}\zeta(3,N) &> \frac{1}{N^3}+\frac{1}{2}\sum_{i\geq N+1}\left(\frac{1}{i^2-i+1/2}+\frac{1}{i^2+i+1/2}\right)\\ &= \frac{1}{N^3}+\frac{1}{2(N^2+N+1/2)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\eta_N &< \frac{1}{2N(\sqrt{N^2+1}-1)}+N\left(\frac{1}{N^2}+\frac{2}{2N+1}\right)-1\\ &< \frac{1}{2N(N-1)}+\frac{1}{N}-\frac{1}{2N+1},\end{aligned}$$

and

$$\begin{aligned}\eta_N &> \frac{1}{N^3}+\frac{1}{2(N^2+N+1/2)}+\frac{2N}{\sqrt{4N^2+1}-1}-1\\ &> \frac{1}{N^3}+\frac{1}{2(N^2+N+1/2)}+\frac{2N+1}{2N}-1\\ &= \frac{1}{N^3}+\frac{1}{2(N^2+N+1/2)}+\frac{1}{2N} > \frac{1}{N^3}+\frac{1}{2N(N+1)}+\frac{1}{2N}.\end{aligned}$$

For example, we have

N	Lower bound of η_N	Upper bound of η_N
2	0.4583333333333333	0.55
10	0.055545454545454544	0.05793650793650794
100	0.00505050495049505	0.0050753806723955975
500	0.001002004007984032	0.001003003009015033
1000	0.0005005005004995005	0.0005007503755629693
5000	0.00010002000400079984	0.00010003000300090015
10000	0.00005000500050004999	0.00005007500375056254

■

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Acta Math. Hungar., **160** (**2**) (2020), 292–313
https://doi.org/10.1007/s10474-019-00974-x
First published online September 4, 2019

A DEPENDENCE WITH COMPLETE CONNECTIONS APPROACH TO GENERALIZED RÉNYI CONTINUED FRACTIONS

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(Received March 9, 2019; revised May 15, 2019; accepted May 16, 2019)

Abstract. We introduce and study in detail a special class of backward continued fractions that represents a generalization of Rényi continued fractions. We investigate the main metrical properties of the digits occurring in these expansions and we construct the natural extension for the transformation that generates the Rényi-type expansion. We also define the random system with complete connections associated with the underlying dynamical system whose ergodic behaviour allows us to prove a variant of Gauss–Kuzmin-type theorem.

1. Introduction

In [6], Gröchenig and Haas investigated u-backward continued fractions associated with the one parameter family of interval maps of the form

$$T_u(x) := \frac{1}{u(1-x)} - \left\lfloor \frac{1}{u(1-x)} \right\rfloor,$$

where $u > 0$, $x \in [0,1)$ and $\lfloor \cdot \rfloor$ denotes the floor function. As the parameter varies the maps T_u exhibit a curious dynamical behaviour. It was known that varying u in the interval $(0,4)$ gives a viable theory of u-backward continued

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Key words and phrases: Rényi continued fraction, Perron–Frobenius operator, random system with complete connections, Gauss–Kuzmin problem.

Mathematics Subject Classification: 11J70, 28D05, 37A30, 60A10.

fractions, which fails when $u \geq 4$. So, for a given $u \in (0,4)$, each irrational $x \in [0,1)$ can be uniquely represented as an infinite continued fraction

$$x = 1 - \cfrac{1}{un_1 - \cfrac{1}{n_2 - \cfrac{1}{un_3 - \ddots - \cfrac{1}{s_k n_k - \ddots}}}} := [a_1, a_2, a_3, \ldots]_u, \tag{1.1}$$

where the integers $n_i = 1 + a_i \geq 2$ and the "coefficient" s_j of n_j alternates between $s_j = 1$ for even j and $s_j = u$ for odd j. The case $u = 1$ was studied by Rényi [17] and provides an alternate approach to continued fractions and rational approximation. The graph of T_1 can be obtained from the graph of the regular (Gauss) continued fraction transformation $G(x) := \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ by reflecting about the line $x = \frac{1}{2}$.

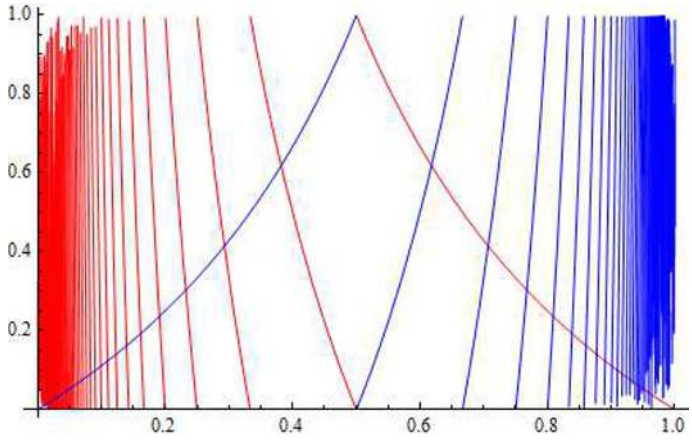

Fig. 1: Graphs of Gauss and Rényi transformations

From the dynamical systems point of view interest in the Rényi map steams from two facts. First, it has a parabolic fixed point, $T_1(0) = 0$ and $T_1'(0) = 1$ and therefore it exhibits non-uniformly hyperbolic behaviour. Secondly, it is semi-conjugate to the full-shift on a countable alphabet, and in particular it has infinite entropy. Therefore, the Rényi map exhibits two sources of complications that are of independent interest and provides an example where we have the opportunity to study how they interact. These features have attracted attention and interesting results have been obtained describing the ergodic theory of this map. In particular Adler and Flatto [1]

showed that there exists an infinite invariant measure absolutely continuous with respect to the Lebesgue measure. There is no finite one.

The main purpose of Gröchenig and Haas was to find an explicit form for the absolutely continuous invariant measure for T_u similar to that of the Gauss measure $\frac{dx}{x+1}$ for G and Rényi's measure $\frac{dx}{x}$ for T_1. While the Gauss measure is finite, the Rényi's measure is infinite. They showed that the invariant measure for T_u is finite if and only if $0 < u < 4$ and $u = u_q \neq 4\cos^2\frac{\pi}{q}$, $q = 3, 4, \ldots$. They also identified the invariant probability measure for T_u corresponding to the values $u = 1/N$ for positive integers $N \geq 2$. This is a finite measure. In this particular case we will call the continued fraction in (1.1) a *Rényi-type continued fraction*. We will indicate $T_{\frac{1}{N}}$ by R_N and we will call it a *Rényi-type (continued fraction) transformation*.

We recall that the use of the ergodicity of G under the Gauss measure in proving ergodic properties of the regular continued fraction expansion was initiated by Doeblin [5]. See also [18] for other important early work.

Several related families of transformations have been investigated with regard to their dynamical properties as well as to the associated continued fraction theories [4,7,8,15]. This has lead to a wide range of connections between continued fractions, diophantine approximation, ergodic theory and hyperbolic geometry [1,13,22].

Our goal in this paper is to start an approach to the metrical theory of Rényi-type continued fraction expansions via dependence with complete connections. Using the natural extensions for the Rényi-type transformations, we give an infinite-order-chain representation of the sequence of the incomplete quotients of these expansions. We then show that the associated random systems with complete connections are with contraction and their transfer operators are regular with respect to the Banach space of Lipschitz functions. Further this leads, in Section 6, to a solution of the Gauss–Kuzmin problem for these continued fractions which allows us to obtain a convergence rate result.

2. Rényi-type continued fraction expansions as dynamical system

For a fixed integer $N \geq 2$, we define the Rényi-type continued fraction transformation $R_N\colon [0,1] \to [0,1]$ by

$$R_N(x) := \begin{cases} \frac{N}{1-x} - \left\lfloor \frac{N}{1-x} \right\rfloor, & x \in [0,1) \\ 0, & x = 1. \end{cases} \tag{2.1}$$

To this transformation, we associate the digits $a_n(x)$, $n \in \mathbb{N}_+ := \{1, 2, 3, \ldots\}$, which are defined by

$$a_n := a_n(x) = a_1(R_N^{n-1}(x)), \quad n \geq 2, \tag{2.2}$$

with $R_N^0(x) = x$ and

$$a_1 := a_1(x) = \begin{cases} \lfloor \frac{N}{1-x} \rfloor & \text{if } x \neq 1, \\ \infty & \text{if } x = 1. \end{cases} \tag{2.3}$$

Putting $\Lambda := \{N, N+1, \ldots\}$, observe that $a_n \in \Lambda$ for any $n \in \mathbb{N}_+$. Using these digits we can write

$$R_N(x) := \frac{N}{1-x} - a_1(x). \tag{2.4}$$

Then

$$x = 1 - \cfrac{N}{1 + a_1 - \cfrac{N}{1 + a_2 - \cfrac{N}{1 + a_3 - \ddots}}} := [a_1, a_2, a_3, \ldots]_R. \tag{2.5}$$

The Rényi-type continued fraction in (2.5) can be viewed as a measure preserving dynamical system $\big([0,1], \mathcal{B}_{[0,1]}, R_N, \rho_N\big)$, where $\mathcal{B}_{[0,1]}$ denotes the σ-algebra of all Borel subsets of $[0,1]$, and

$$\rho_N(A) := \frac{1}{\log(\frac{N}{N-1})} \int_A \frac{\mathrm{d}x}{x+N-1}, \quad A \in \mathcal{B}_{[0,1]} \tag{2.6}$$

is the invariant probability measure under R_N [6].

Let $\frac{p_n}{q_n} := \frac{p_n(x)}{q_n(x)}$ denote the partial fractions obtained by applying the transformation R_N n-times, so

$$\frac{p_n}{q_n} = 1 - \cfrac{N}{1 + a_1 - \cfrac{N}{1 + a_2 - \ddots - \cfrac{N}{1 + a_n}}}. \tag{2.7}$$

For the functions p_n and q_n we obtain the following recurrence relations:

$$p_0 = 1, \quad p_1 = 1 + a_1 - N, \quad p_n = (1 + a_n)p_{n-1} - Np_{n-2}, \ n \geq 2, \tag{2.8}$$

$$q_0 = 1, \quad q_1 = 1 + a_1, \quad q_n = (1 + a_n)q_{n-1} - Nq_{n-2}, \ n \geq 2. \tag{2.9}$$

Using these recurrences, induction easily gives that

$$p_{n-1}q_n - p_nq_{n-1} = N^n, \quad n \in \mathbb{N}_+ \tag{2.10}$$

and

$$x = \frac{p_n + (R_N^n - 1)p_{n-1}}{q_n + (R_N^n - 1)q_{n-1}}, \quad n \in \mathbb{N}_+. \tag{2.11}$$

We obtain

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{N^n}{q_n(q_n - q_{n-1})}, \quad n \in \mathbb{N}_+. \tag{2.12}$$

3. The probabilistic structure of $(a_n)_{n\in\mathbb{N}_+}$ under the Lebesgue measure

We start by defining the n-th order cylinder associated to the digits $(a_n)_{n\in\mathbb{N}_+}$ of the Rényi-type continued fraction (2.4). An n-block $(a_1, a_2, \ldots, a_n)$ is said to be *admissible* for the expansion in (2.4) if there exists $x \in [0, 1)$ such that $a_i(x) = a_i$ for all $1 \leq i \leq n$. If $(a_1, a_2, \ldots, a_n)$ is an admissible sequence, we call the set

$$I(a_1, a_2, \ldots, a_n) = \big\{x \in [0, 1] : a_1(x) = a_1, a_2(x) = a_2, \ldots, a_n(x) = a_n\big\}, \tag{3.1}$$

the n-th order cylinder. As we mentioned above, $(a_1, a_2, \ldots, a_n) \in \Lambda^n$. For example, for any $a_1 = i \in \Lambda$ we have

$$I(a_1) = \big\{x \in [0, 1] : a_1(x) = a_1\big\} = \Big[1 - \frac{N}{i}, 1 - \frac{N}{i+1}\Big). \tag{3.2}$$

By induction, it can be shown that

$$I(a_1, a_2, \ldots, a_n) = \Big[\frac{p_n - p_{n-1}}{q_n - q_{n-1}}, \frac{p_n}{q_n}\Big), \tag{3.3}$$

for all $n \geq 1$. Hence, using (2.10) we have

$$\lambda\left(I(a_1, a_2, \ldots, a_n)\right) = \frac{N^n}{q_n(q_n - q_{n-1})}, \tag{3.4}$$

where λ is the Lebesgue measure and (q_n) is as in (2.9).

To derive the so-called Brodén–Borel–Lévy formula [9,10], let us define $(s_n)_{n\in\mathbb{N}}$ by

$$s_0 := 1, \quad s_n := 1 - N\frac{q_{n-1}}{q_n}. \tag{3.5}$$

From (2.9), $s_n = 1 - N/(a_n + s_{n-1})$. Hence,

$$s_n = 1 - \cfrac{N}{1 + a_n - \cfrac{N}{1 + a_{n-1} - \ddots - \cfrac{N}{1 + a_1}}} = [a_n, a_{n-1}, \dots, a_1]_R. \tag{3.6}$$

PROPOSITION 3.1 (Brodén–Borel–Lévy-type formula). *Let λ denote Lebesgue measure on $[0,1]$. For any $n \in \mathbb{N}_+$, the conditional probability $\lambda(R_N^n < x \mid a_1, \dots, a_n)$ is given by*

$$\lambda\big(R_N^n < x \mid a_1, \dots, a_n\big) = \frac{Nx}{N - (1-x)(1-s_n)}, \quad x \in [0,1] \tag{3.7}$$

where (s_n) is as in (3.5) *and $a_1, \dots, a_n$ are as in* (2.2) *and* (2.3).

PROOF. By definition, we have

$$\lambda(R_N^n < x \mid a_1, \dots, a_n) = \frac{\lambda((R_N^n < x) \cap I(a_1, \dots, a_n))}{\lambda(I(a_1, \dots, a_n))}$$

for any $n \in \mathbb{N}_+$ and $x \in [0,1]$. From (2.11) and (3.3) we have

$$\begin{aligned}\lambda\big((R_N^n < x) \cap I(a_1, \dots, a_n)\big) &= \Big|\frac{p_n + (x-1)p_{n-1}}{q_n + (x-1)q_{n-1}} - \frac{p_n - p_{n-1}}{q_n - q_{n-1}}\Big| \\ &= \frac{N^n x}{(q_n - q_{n-1})(q_n + (x-1)q_{n-1})}.\end{aligned}$$

From this and (3.4), we have that

$$\begin{aligned}\lambda\big(R_N^n < x \mid a_1, \dots, a_n\big) &= \frac{\lambda((R_N^n < x) \cap I(a_1, \dots, a_n))}{\lambda(I(a_1, \dots, a_n))} \\ &= \frac{xq_n}{q_n + (x-1)q_{n-1}} = \frac{Nx}{N - (1-x)(1-s_n)}\end{aligned}$$

for any $n \in \mathbb{N}_+$ and $x \in [0,1]$. □

The Brodén–Borel–Lévy-type formula allows us to determine the probabilistic structure of the digits $(a_n)_{n \in \mathbb{N}_+}$ under λ.

PROPOSITION 3.2. *For any $i \geq N$ and $n \in \mathbb{N}_+$, we have*

$$\lambda(a_1 = i) = \frac{N}{i(i+1)}, \quad \lambda(a_{n+1} = i | a_1, \dots, a_n) = P_{N,i}(s_n) \tag{3.8}$$

where (s_n) *is as in* (3.5), *and*

$$P_{N,i}(x) := \frac{x+N-1}{(x+i)\,(x+i-1)}. \tag{3.9}$$

PROOF. From (3.2), the case $\lambda(a_1 = i)$ holds. For $n \in \mathbb{N}_+$ and $x \in [0,1]$, we have $R_N^n(x) = [a_{n+1}, a_{n+2}, \ldots]_R$, where (a_n) is as in (2.2). By using (3.7), we have

$$\begin{aligned}\lambda(a_{n+1} = i \mid a_1, \ldots, a_n) &= \lambda\Big(R_N^n \in \Big[1 - \frac{N}{i}, 1 - \frac{N}{i+1}\Big) \,\Big|\, a_1, \ldots, a_n\Big) \\ &= \frac{N(1-\frac{N}{i+1})}{N - \frac{N}{i+1}(1-s_n)} - \frac{N(1-\frac{N}{i})}{N - \frac{N}{i}(1-s_n)} = P_{N,i}(s_n). \quad \square\end{aligned}$$

It is easy to check that

$$\sum_{i=N}^{\infty} P_{N,i}(x) = 1 \quad \text{for any } x \in [0,1]. \tag{3.10}$$

REMARK 3.3. Proposition 3.2 is the starting point of an approach to the metrical theory of Rényi-type continued fraction expansions via dependence with complete connections (see [9, Section 5.2]). We apply this method in Section 6 to obtain a solution of Gauss–Kuzmin-type problem for Rényi-type continued fraction expansions.

COROLLARY 3.4. *The sequence* $(s_n)_{n\in\mathbb{N}_+}$ *with* $s_0 = 1$ *is a homogeneous* $[0,1]$*-valued Markov chain on* $\big([0,1], \mathcal{B}_{[0,1]}, \lambda\big)$ *with the following transition mechanism*: *from state* $s \in [0,1]$ *the only possible one-step transitions are those of states* $1 - N/(s+i)$, $i \geq N$, *with corresponding probabilities* $P_{N,i}(s)$, $i \geq N$.

4. Natural extension and extended random variables

Fix an integer $N \geq 2$. In this section, we introduce the natural extension $\overline{R}_N$ of R_N in (2.1) and its extended random variables [10].

4.1. Natural extension. Let $\big([0,1], \mathcal{B}_{[0,1]}, R_N\big)$ be as in Section 2. Define $(u_{N,i})_{i\geq N}$ by

$$u_{N,i}\colon [0,1] \to [0,1]; \quad u_{N,i}(x) := 1 - \frac{N}{x+i}, \quad x \in [0,1]. \tag{4.1}$$

For each $i \geq N$, $u_{N,i}$ is a right inverse of R_N, that is,

$$(R_N \circ u_{N,i})(x) = x, \quad \text{for any } x \in [0,1]. \tag{4.2}$$

Furthermore, if $a_1(x) = i$, then $(u_{N,i} \circ R_N)(x) = x$ where a_1 is as in (2.3).

DEFINITION 4.1. The natural extension $([0,1]^2, \mathcal{B}_{[0,1]^2}, \overline{R}_N)$ of $\big([0,1], \mathcal{B}_{[0,1]}, R_N\big)$ is the transformation $\overline{R}_N$ of the square space $([0,1]^2, \mathcal{B}^2_{[0,1]}) := ([0,1], \mathcal{B}_{[0,1]}) \times ([0,1], \mathcal{B}_{[0,1]})$ defined as follows [16]:

$$\overline{R}_N : [0,1]^2 \to [0,1]^2;$$

$$\overline{R}_N(x,y) := (R_N(x), u_{N,a_1(x)}(y)), \quad (x,y) \in [0,1]^2. \tag{4.3}$$

From (4.2), we see that $\overline{R}_N$ is bijective on $[0,1]^2$ with the inverse

$$(\overline{R}_N)^{-1}(x,y) = (u_{N,a_1(y)}(x), R_N(y)), \quad (x,y) \in [0,1]^2. \tag{4.4}$$

Iterations of (4.1) and (4.4) are given as follows for each $n \geq 2$:

$$(\overline{R}_N)^n(x,y) = \big(R_N^n(x),\, [a_n(x), a_{n-1}(x), \ldots, a_2(x), a_1(x) + y - 1]_R\big),$$

$$(\overline{R}_N)^{-n}(x,y) = \big([a_n(y), a_{n-1}(y), \ldots, a_2(y), a_1(y) + x - 1]_R, R_N^n(y)\big).$$

For ρ_N in (2.6), we define its *extended measure* $\overline{\rho}_N$ on $\big([0,1]^2, \mathcal{B}^2_{[0,1]}\big)$ as

$$\overline{\rho}_N(B) := \frac{1}{\log(\frac{N}{N-1})} \iint_B \frac{N \,\mathrm{d}x \,\mathrm{d}y}{\{N - (1-x)(1-y)\}^2}, \quad B \in \mathcal{B}^2_{[0,1]}. \tag{4.5}$$

Then $\overline{\rho}_N(A \times [0,1]) = \overline{\rho}_N([0,1] \times A) = \rho_N(A)$ for any $A \in \mathcal{B}_{[0,1]}$. The measure $\overline{\rho}_N$ is preserved by $\overline{R}_N$, i.e., $\overline{\rho}_N((\overline{R}_N)^{-1}(B)) = \overline{\rho}_N(B)$ for any $B \in \mathcal{B}^2_{[0,1]}$.

4.2. Extended random variables. Define the projection $E\colon [0,1]^2 \to [0,1]$ by $E(x,y) := x$. With respect to $\overline{R}_N$ in (4.1), define *extended incomplete quotients* $\overline{a}_l(x,y)$, $l \in \mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ at $(x,y) \in [0,1]^2$ by

$$\overline{a}_l(x,y) := (a_1 \circ E)\big((\overline{R}_N)^{l-1}(x,y)\big), \quad l \in \mathbb{Z}. \tag{4.6}$$

REMARK 4.2. (i) Note that $\overline{a}_l(x,y)$ in (4.6) is also well-defined for $l \leq 0$ because $\overline{R}_N$ is invertible. By (4.1) and (4.4) we have

$$\overline{a}_n(x,y) = a_n(x), \quad \overline{a}_0(x,y) = a_1(y), \quad \overline{a}_{-n}(x,y) = a_{n+1}(y), \tag{4.7}$$

for any $n \in \mathbb{N}_+$ and $(x,y) \in [0,1]^2$.

(ii) Since the measure $\overline{\rho}_N$ is preserved by $\overline{R}_N$, the doubly infinite sequence $(\overline{a}_l(x,y))_{l \in \mathbb{Z}}$ is strictly stationary (i.e., its distribution is invariant under a shift of the indices) under $\overline{\rho}_N$.

THEOREM 4.3. *Fix* $(x,y)\in[0,1]^2$ *and let* $\overline{a}_l:=\overline{a}_l(x,y)$ *for* $l\in\mathbb{Z}$. *Set* $a:=[\overline{a}_0,\overline{a}_{-1},\ldots]_R$. *Then the following holds for any* $x\in[0,1]$:

$$\overline{\rho}_N\big([0,x]\times[0,1]\mid\overline{a}_0,\overline{a}_{-1},\ldots\big)=\frac{Nx}{N-(1-x)(1-a)}\quad\overline{\rho}_N\text{-a.s.}\tag{4.8}$$

PROOF. Recall the cylinder in (3.1). Let I_n denote the cylinder $I(\overline{a}_0,\overline{a}_{-1},\ldots,\overline{a}_{-n})$ for $n\in\mathbb{N}$. We have

$$\overline{\rho}_N\big([0,x]\times[0,1]\mid\overline{a}_0,\overline{a}_{-1},\ldots\big)=\lim_{n\to\infty}\overline{\rho}_N\big([0,x]\times[0,1]\mid\overline{a}_0,\ldots,\overline{a}_{-n}\big)\ \overline{\rho}_N\text{-a.s.}$$

and

$$\begin{aligned}\overline{\rho}_N\big([0,x]\times[0,1]\mid\overline{a}_0,\ldots,\overline{a}_{-n}\big)&=\frac{\overline{\rho}_N([0,x]\times I_n)}{\overline{\rho}_N([0,1)\times I_n)}\\&=\frac{1}{\log(\frac{N}{N-1})}\,\frac{1}{\rho_N(I_n)}\int_{I_n}\mathrm{d}y\int_0^x\frac{N\,\mathrm{d}u}{\{N-(1-u)(1-y)\}^2}\\&=\frac{1}{\rho_N(I_n)}\int_{I_n}\frac{Nx}{\{N-(1-x)(1-y)\}^2}\,\rho_N(\mathrm{d}y)=\frac{Nx}{\{N-(1-x)(1-y_n)\}^2},\end{aligned}$$

for some $y_n\in I_n$. Since

$$\lim_{n\to\infty}y_n=[\overline{a}_0,\overline{a}_{-1},\ldots]_R=a,\tag{4.9}$$

the proof is completed. □

The stochastic property of $(\overline{a}_l)_{l\in\mathbb{Z}}$ under $\overline{\rho}_N$ is as follows.

COROLLARY 4.4. *For any* $i\geq N$, *we have*

$$\overline{\rho}_N\big(\overline{a}_1=i\mid\overline{a}_0,\overline{a}_{-1},\ldots\big)=P_{N,i}(a)\quad\overline{\rho}_N\text{-a.s.}\tag{4.10}$$

where $a=[\overline{a}_0,\overline{a}_{-1},\ldots]_R$ *and* $P_{N,i}$ *is as in* (3.9).

PROOF. Let I_n be as in the proof of Theorem 4.3. We have

$$\overline{\rho}_N\big(\overline{a}_1=i\mid\overline{a}_0,\overline{a}_{-1},\ldots\big)=\lim_{n\to\infty}\overline{\rho}_N\big(\overline{a}_1=i\mid I_n\big).\tag{4.11}$$

Also

$$(\overline{a}_1=i)=\Big[1-\frac{N}{i},1-\frac{N}{i+1}\Big)\times[0,1],\quad i\geq N.\tag{4.12}$$

Now

$$\overline{\rho_N}\Big(\Big[1-\frac{N}{i},1-\frac{N}{i+1}\Big)\times[0,1]\ \Big|\ I_n\Big)=\frac{\overline{\rho}_N([1-\frac{N}{i},1-\frac{N}{i+1})\times I_n)}{\overline{\rho}_N([0,1]\times I_n)}$$

$$= \frac{1}{\rho_N(I_n)} \int_{I_n} P_{N,i}(y)\, \rho_N(\mathrm{d}y) = P_{N,i}(y_n),$$

for some $y_n \in I_n$. Given (4.9), the proof is completed. □

REMARK 4.5. The strict stationarity of $(\overline{a}_l)_{l\in\mathbb{Z}}$, under $\overline{\rho}_N$ implies that

$$\overline{\rho}_N\big(\overline{a}_{l+1} = i \mid \overline{a}_l, \overline{a}_{l-1}, \dots\big) = P_{N,i}(a) \quad \overline{\rho}_N\text{-a.s.} \tag{4.13}$$

for any $i \geq N$ and $l \in \mathbb{Z}$, where $a = [\overline{a}_l, \overline{a}_{l-1}, \dots]_R$. The equation (4.13) emphasizes that $(\overline{a}_l)_{l\in\mathbb{Z}}$ is a chain of infinite order in the theory of dependence with complete connections [9].

Define extended random variables $(\overline{s}_l)_{l\in\mathbb{Z}}$ by $\overline{s}_l := [\overline{a}_l, \overline{a}_{l-1}, \dots]_R$, $l \in \mathbb{Z}$. Clearly, $\overline{s}_l = \overline{s}_0 \circ (\overline{R}_N)^l$, $l \in \mathbb{Z}$. It follows from Corollary 4.4 that $(\overline{s}_l)_{l\in\mathbb{Z}}$ is a strictly stationary $[0,1)$-valued Markov process on $\big([0,1]^2, \mathcal{B}^2_{[0,1]}, \overline{\rho}_N\big)$ with the following transition mechanism. From state $\overline{s} \in [0,1]$ the possible transitions are to any state $1 - N/(\overline{s}+i)$ with corresponding transition probability $P_{N,i}(\overline{s})$, $i \geq N$. Clearly, for any $l \in \mathbb{Z}$ we have

$$\overline{\rho}_N(\overline{s}_l < x) = \overline{\rho}_N([0,1] \times [0,x)) = \rho_N([0,x)), \quad x \in [0,1]. \tag{4.14}$$

Motivated by Theorem 4.3, we shall consider the one-parameter family $\{\rho_{N,t} : t \in [0,1]\}$ of (conditional) probability measures on $([0,1], \mathcal{B}_{[0,1]})$ defined by their distribution functions

$$\rho_{N,t}([0,x]) := \frac{Nx}{N - (1-x)(1-t)}, \quad x,t \in [0,1]. \tag{4.15}$$

Note that $\rho_{N,1} = \lambda$.

For any $t \in [0,1]$ put

$$s_{0,t} := t, \quad s_{n,t} := 1 - \frac{N}{a_n + s_{n-1,t}}, \quad n \in \mathbb{N}_+. \tag{4.16}$$

REMARK 4.6. It follows from the properties just described for the process $(\overline{s}_l)_{l\in\mathbb{Z}}$ that the sequence $(s_{n,t})_{n\in\mathbb{N}_+}$ is an $[0,1]$-valued Markov chain on $([0,1], \mathcal{B}_{[0,1]}, \rho_{N,t})$ which starts at $s_{0,t} := t$ and has the following transition mechanism: from state $s \in [0,1]$ the possible transitions are to any state $1 - N/(s+i)$ with corresponding transition probability $P_i(s)$, $i \geq N$.

5. Perron–Frobenius operators

Let $([0,1], \mathcal{B}_{[0,1]}, R_N, \rho_N)$ be as in Section 2. In this section, we derive its Perron–Frobenius operator.

Let μ be a probability measure on $([0,1],\mathcal{B}_{[0,1]})$ such that $\mu(R_N^{-1}(A))=0$ whenever $\mu(A)=0$ for $A\in\mathcal{B}_{[0,1]}$. For example, this condition is satisfied if R_N is μ-preserving, that is, $\mu R_N^{-1}=\mu$. Let $L^1([0,1],\mu):=\left\{f\colon [0,1]\to\mathbb{C}:\int_0^1|f|\,\mathrm{d}\mu<\infty\right\}$. The *Perron–Frobenius operator* of $([0,1],\mathcal{B}_{[0,1]},R_N,\mu)$ can be defined by the Radon–Nikodym theorem as the unique linear and positive operator on the Banach space $L^1([0,1],\mu)$ satisfying:

$$\int_A Uf\,\mathrm{d}\mu=\int_{R_N^{-1}(A)} f\,\mathrm{d}\mu \quad \text{for all } A\in\mathcal{B}_{[0,1]},\ f\in L^1([0,1],\mu). \tag{5.1}$$

For more details, see [3,10].

PROPOSITION 5.1. *Let $([0,1],\mathcal{B}_{[0,1]},R_N,\rho_N)$ be as in Section 2, and let U denote its Perron–Frobenius operator. Then the following holds.*

(i) *We have*

$$Uf(x)=\sum_{i\geq N} P_{N,i}(x)\,f\left(u_{N,i}(x)\right),\quad f\in L^1([0,1],\rho_N), \tag{5.2}$$

where $P_{N,i}$ and $u_{N,i}$ are as in (3.9) *and* (4.1)*, respectively.*

(ii) *Let μ be a probability measure on $([0,1],\mathcal{B}_{[0,1]})$ such that μ is absolutely continuous with respect to the Lebesgue measure λ and let $h:=\mathrm{d}\mu/\mathrm{d}\lambda$ a.e. in $[0,1]$. Then for any $n\in\mathbb{N}_+$ and $A\in\mathcal{B}_{[0,1]}$, we have*

$$\mu\left(R_N^{-n}(A)\right)=\int_A U^n f(x)\,\mathrm{d}\rho_N(x) \tag{5.3}$$

where $f(x):=(\log(\frac{N}{N-1}))(x+N-1)h(x)$ for $x\in[0,1]$.

PROOF. (i) Let $R_{N,i}$ denote the restriction of R_N to the subinterval $I(i):=(1-\frac{N}{i},1-\frac{N}{i+1}]$, $i\geq N$, that is,

$$R_{N,i}(x)=\frac{N}{1-x}-i,\quad x\in I(i). \tag{5.4}$$

Let $C(A):=(R_N)^{-1}(A)$ and $C_i(A):=(R_{N,i})^{-1}(A)$ for $A\in\mathcal{B}_{[0,1]}$. Since $C(A)=\bigcup_i C_i(A)$ and $C_i\cap C_j$ is a null set when $i\neq j$, we have

$$\int_{C(A)} f\,\mathrm{d}\rho_N=\sum_{i\geq N}\int_{C_i(A)} f\,\mathrm{d}\rho_N,\quad f\in L^1([0,1],\rho_N),\ A\in\mathcal{B}_{[0,1]}. \tag{5.5}$$

For any $i \geq N$, by the change of variable $x = (R_{N,i})^{-1}(y) = 1 - \dfrac{N}{y+i}$, we obtain

$$\begin{aligned}(5.6)\qquad \int_{C_i(A)} f(x)\,\rho_N(\mathrm{d}x) &= \Big(\log\Big(\frac{N}{N-1}\Big)\Big)^{-1} \int_{C_i(A)} \frac{f(x)}{x+N-1}\,\mathrm{d}x \\ &= \Big(\log\Big(\frac{N}{N-1}\Big)\Big)^{-1} \int_A \frac{1}{(y+i)(y+i-1)} f(u_{N,i}(y))\,\mathrm{d}y \\ &= \int_A P_{N,i}(y)\, f(u_{N,i}(y))\,\rho_N(\mathrm{d}y).\end{aligned}$$

Now, (5.2) follows from (5.5) and (5.6).

(ii) We will use mathematical induction. For $n = 0$, the equation (5.3) holds by the definitions of f and h. Assume that (5.3) holds for some $n \in \mathbb{N}$. Then

$$(5.7)\qquad \mu\big(R_N^{-(n+1)}(A)\big) = \mu(R_N^{-n}(R_N^{-1}(A))) = \int_{C(A)} U^n f(x)\,\rho_N(\mathrm{d}x),$$

and by definition, we have

$$(5.8)\qquad \int_{C(A)} U^n f(x)\,\rho_N(\mathrm{d}x) = \int_A U^{n+1} f(x)\,\rho_N(\mathrm{d}x).$$

Therefore,

$$(5.9)\qquad \mu\big(R_N^{-(n+1)}(A)\big) = \int_A U^{n+1} f(x)\rho_N(\mathrm{d}x)$$

which ends the proof. □

6. Random systems with complete connections and the Gauss–Kuzmin-type problem

In this section, we introduce random systems with complete connections (RSCC) and describe their properties, i.e. we present the asymptotic and ergodic properties of operators associated with a RSCC. The general results presented will be applied to Rényi-type continued fraction expansions. For more detail, see [9,12].

6.1. Definitions and properties. A RSCC is often called an *iterated function system with place-dependent probabilities* or simply an *iterated function system* (=IFS) [2]. For applications of RSCC to continued fraction expansions, see also [11,14,19,20].

DEFINITION 6.1. A random system with complete connections is a quadruple

$$\{(W, \mathcal{W}), X, u, P\} \tag{6.1}$$

where

(i) $(W, \mathcal{W})$ is a measurable space and X is a non-empty set;

(ii) $u\colon W \times X \to W$ is a measurable function with respect to W for each element in X;

(iii) $P\colon W \times X \to [0,1]$ satisfies $\sum_{x\in X} P(w,x) = 1$ for each $w \in W$, and $P(\cdot, x)$ is a measurable function on $(W, \mathcal{W})$ for each $x \in X$.

The function u induces a unique right action of X^n on W which is denoted by the same symbol u. Here

$$u\colon W \times X^n \to W; \quad (w,x) \mapsto wx := u(w,x). \tag{6.2}$$

For example, $w(xx')$ is defined by $(wx)x' = u(u(w,x),x')$ for $w \in W$, $x \in X^n$ and $x' \in X$. For $x = x_1 \cdots x_n \in X^n$, we can write $wx := w(x_1 \cdots x_n)$ for $w \in W$.

Let $\mathcal{P}(X)$ denote the power set of X. The domain $W \times X$ of P is extended to $W \times \mathcal{P}(X)$ by setting

$$P(w,A) := \begin{cases} \sum_{x\in A} P(w,x), & A \neq \emptyset, \\ 0, & A = \emptyset, \end{cases} \quad (w,A) \in W \times \mathcal{P}(X). \tag{6.3}$$

From Definition 6.1(iii), we see that $(X, \mathcal{P}(X), P(w,\cdot))$ is a probability space for each $w \in W$.

As a generalization of P, we define $P_r\colon W \times X^r \to [0,1]$ for each $r \geq 1$ by

$$P_1 := P, \quad P_r(w, xx') := P_{r-1}(w,x)P(wx,x'), \tag{6.4}$$

for $w \in W$, $x \in X^{r-1}$, $x \in X$, $r \geq 2$, where the notation in (6.2) is used.

For $A \subset X^r$, we also define

$$P_r(w,A) := \begin{cases} \sum_{x\in A} P_r(w,x) & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases} \tag{6.5}$$

For $w, w' \in W$, define the subset $X(w,w')$ of X by

$$X(w,w') := \{x \in X : u(w,x) = w'\}. \tag{6.6}$$

By using the extension of P in (6.3) and $X(w,w')$ in (6.6), define the new function $Q\colon W \times W \to [0,1]$ by

$$Q(w,w') := P(w, X(w,w')), \quad (w,w') \in W \times W. \tag{6.7}$$

We extend the domain of Q to $W \times \mathcal{W}$ by setting

$$Q(w,B) := P(w, X(w,B)) = \sum_{x \in X(w,B)} P(w,x), \quad (w,B) \in W \times \mathcal{W}. \tag{6.8}$$

We call Q the *((W,W)-) transition probability function* associated with $\{(W,\mathcal{W}), X, u, P\}$.

Let $L^\infty(W)$ denote the Banach space of complex-valued bounded measurable functions defined on $(W,\mathcal{W})$. In the following, W will be a measurable subset of the measurable space $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ and $\mathcal{W} = \mathcal{B}_W$. We can therefore use the absolute value $|\cdot|$ and Lebesgue measure on W induced by $\mathbb{R}$.

The *transfer operator* U on $L^\infty(W)$ is defined by

$$Uf(w) := \sum_{x \in X} P(w,x)\, f(wx), \quad f \in L^\infty(W),\ w \in W. \tag{6.9}$$

Also for each $n \geq 1$, note that

$$U^n f(w) = \sum_{x \in X^n} P_n(w,x) f(wx)$$

where U^n denotes the n-the iterate of U and P_n is as in (6.4) for $r = n$.

It follows that the n-step transition probability function is given by

$$Q^n(w,B) = P_n\big(w, B_w^{(n)}\big), \quad w \in W,\ B \in \mathcal{W}, n \in \mathbb{N}_+, \tag{6.10}$$

where $B_w^{(n)} = \{x^{(n)} : wx^{(n)} \in B\}$.

Equivalently the transfer operator U can be defined by

$$Uf(w) := \int_W f(w') Q(w, \mathrm{d}w'), \quad f \in B(W,\mathcal{W}). \tag{6.11}$$

Its iterates are given by

$$U^n f(w) = \int_W f(w') Q^n(w, \mathrm{d}w'), \quad n \in \mathbb{N}_+, \tag{6.12}$$

where Q^n is the n-step transition probability function.

Now define

$$Q_n(w,B) = \frac{1}{n} \sum_{k=1}^{n} Q^k(w,B), \quad (w,B) \in W \times \mathcal{W},\ n \geq 1.$$

So $Q_1 = Q$. Then $(W, \mathcal{W}, Q_n(w,\cdot))$ is also a probability space for each $w \in W$. Let U_n be the Markov operator associated to Q_n, i.e.,

$$U_n f(w) := \int_W f(w') Q_n(w, \mathrm{d}w'), \quad f \in B(W, \mathcal{W}). \tag{6.13}$$

Next, let $L(W)$ denote the Banach space of all complex-valued Lipschitz continuous functions on W with the following norm $\|\cdot\|_L$:

$$\|f\|_L := \|f\| + s(f), \tag{6.14}$$

where

$$\|f\| := \sup_{w \in W} |f(w)|, \quad s(f) := \sup_{w' \neq w''} \frac{|f(w') - f(w'')|}{|w' - w''|}, \quad f \in L(W). \tag{6.15}$$

Following [9], we introduce several characterizations of the operator U in (6.9) as follows.

Definition 6.2. Let W, U, U_n, $L(W)$ be as in (6.1), (6.9), (6.13) and (6.14), respectively. We say

(i) U is ordered if there exists a bounded linear operator U^∞ on $L(W)$ such that

$$\lim_{n\to\infty} \|U_n f - U^\infty f\|_L = 0, \quad f \in L(W); \tag{6.16}$$

(ii) U is aperiodic if there exists a bounded linear operator U^∞ on $L(W)$ such that

$$\lim_{n\to\infty} \|U^n f - U^\infty f\|_L = 0, \quad f \in L(W), \tag{6.17}$$

where U^n is the n-th iterate of U for $n \geq 1$;

(iii) U is ergodic with respect to $L(W)$ if U is ordered and the rank of U^∞ in (6.16) is 1;

(iv) U is regular with respect to $L(W)$ if U is ergodic and aperiodic;

(v) the Markov chain corresponding to U is regular if U is regular with respect to $L(W)$, and

(vi) the Markov chain corresponding to U is ordered if U is ordered with respect to $L(W)$.

Remark 6.3. Definition 6.2(ii) means that the sequence (U^n) converges to the operator U^∞ with respect to the strong operator topology on $L(W)$.

Definition 6.4. The transfer operator U of a Markov chain with state space W is said to be a Doeblin–Fortet operator if and only if U maps $L(W)$

into $L(W)$ boundedly with respect to $\|\cdot\|_L$ and there exist $k \in \mathbb{N}_+$, $\alpha \in [0,1)$ and $\beta < \infty$ such that

$$s(U^k f) \le \alpha \cdot s(f) + \beta \cdot \|f\|, \quad f \in L(W). \tag{6.18}$$

Alternatively, the Markov chain itself is said to be a *Doeblin–Fortet chain*.

DEFINITION 6.5. A Markov chain is said to be compact if and only if its state space is a compact metric space (W, dist) and its transfer operator is a Doeblin–Fortet operator.

THEOREM 6.6. *The Markov chain associated with a RSCC with contraction is a Doeblin–Fortet chain.*

THEOREM 6.7. *A compact Markov chain is ordered with respect to $L(W)$ and there exists a transition probability function Q^∞ on $(W, \mathcal{W})$ such that*

$$U^\infty f = \int_W f(w) Q^\infty(\mathrm{d}w), \quad f \in L(W). \tag{6.19}$$

Moreover, for any $w \in W$, Q^∞ is a stationary probability for the chain.

LEMMA 6.8. *Assume that the Markov operator U is aperiodic with respect to $L(W)$. Then we have $U^\infty U^\infty = U^\infty$, $(U - U^\infty)U^\infty = U^\infty(U - U^\infty) = 0$ and $U^n = U^\infty + (U - U^\infty)^n$ for $n \in \mathbb{N}_+$. Moreover, there exist positive constants $q < 1$ and k such that*

$$\left\| U^n f - U^\infty f \right\|_L \le k q^n \|f\|_L, \quad f \in L(W), n \in \mathbb{N}_+. \tag{6.20}$$

DEFINITION 6.9. The system $\{(W, \mathcal{W}), X, u, P\}$ is a RSCC with contraction if the following conditions are satisfied

(i) $\vartheta_1 < \infty$;
(ii) $\vartheta_\ell < 1$ for some $\ell \ge 1$, and
(iii) $\Upsilon < \infty$,

where (ϑ_k) and Υ are defined as

$$\vartheta_k := \sup_{w' \ne w''} \sum_{x \in X^k} P(w', x) \frac{|w'x - w''x|}{|w' - w''|}, \quad k \ge 1, \tag{6.21}$$

$$\Upsilon := \sup_{A \subset X} \sup_{w' \ne w''} \frac{|P(w', A) - P(w'', A)|}{|w' - w''|}. \tag{6.22}$$

THEOREM 6.10. *Let $\{(W, \mathcal{W}), X, u, P\}$ be a RSCC with contraction. For Q^n in* (6.10)*, define (σ_n) by*

$$\sigma_n(w) := \operatorname{supp} Q^n(w, \cdot), \quad w \in W \tag{6.23}$$

where $\operatorname{supp}\mu$ *denotes the support of a measure* μ. *Assume that* W *is compact. Then the following holds.*

(i) *The Markov chain associated with the RSCC is regular if and only if there exists a point* $w_0 \in W$ *such that*

$$\lim_{n\to\infty} \operatorname{dist}(\sigma_n(w), w_0) = 0 \quad \text{for all } w \in W \tag{6.24}$$

where $\operatorname{dist}(A, w) := \inf_{w'\in A} |w' - w|$ *for* $A \subset W$.

(ii) *For* (σ_n) *in* (6.23), *we have*

$$\sigma_{m+n}(w) = \overline{\bigcup_{w'\in\sigma_m(w)} \sigma_n(w')}, \tag{6.25}$$

for all m, $n \in \mathbb{N}_+$, $w \in W$. *Here the overline means the topological closure in* W.

THEOREM 6.11. *Let* $\{(W, \mathcal{W}), X, u, P\}$ *be a RSCC with contraction such that* W *is compact and assume that* $\{(W, \mathcal{W}), X, u, P\}$ *has a regular associated Markov chain. Then* $\{(W, \mathcal{W}), X, u, P\}$ *is uniformly ergodic.*

6.2. The Gauss–Kuzmin-type theorem for the Rényi-type continued fraction expansions. In what follows we give the RSCC associated with the dynamical system $\big([0,1], \mathcal{B}_{[0,1]}, R_N\big)$ where R_N is the Rényi-type continued fraction expansions. The ergodic behaviour of this RSCC gives us the asymptotic behaviour of R_N^{-n} as $n \to \infty$ that represents the Gauss–Kuzmin-type problem for Rényi-type continued fraction expansions.

Proposition 3.2 leads us to the RSCC $\big\{([0,1], \mathcal{B}_{[0,1]}), \Lambda, u, P\big\}$ as follows:

$$\begin{cases} u\colon [0,1]\times\Lambda \to [0,1]; \quad u(x,i) := u_{N,i}(x) = 1 - N/(x+i), \\ P\colon [0,1]\times\Lambda \to [0,1]; \quad P(x,i) := P_{N,i}(x) = \dfrac{x+N-1}{(x+i)(x+i-1)}, \\ \Lambda = \{N, N+1, \ldots\}. \end{cases} \tag{6.26}$$

We showed in Section 4.1 that $u(\cdot, i)$ is the right inverse of R_N for each $x \in [0,1]$. This shows that the dynamical system $([0,1], \mathcal{B}_{[0,1]}, R_N)$ is encoded by the RSCC $\big\{([0,1], \mathcal{B}_{[0,1]}), \Lambda, u, P\big\}$.

LEMMA 6.12. *The system* $\big\{([0,1], \mathcal{B}_{[0,1]}), \Lambda, u, P\big\}$ *is a RSCC with contraction.*

PROOF. We have

$$\frac{\mathrm{d}}{\mathrm{d}x} u(x,i) = \frac{\mathrm{d}}{\mathrm{d}x} u_{N,i}(x) = \frac{N}{(x+i)^2},$$

$$\frac{\mathrm{d}}{\mathrm{d}x}P(x,i)=\frac{\mathrm{d}}{\mathrm{d}x}P_{N,i}(x)=\frac{i-N}{(x+i-1)^2}-\frac{i+1-N}{(x+i)^2}$$

for any $x\in[0,1]$ and $i\in\Lambda$. Thus,

$$\sup_{x\in[0,1]}\left|\frac{\mathrm{d}}{\mathrm{d}x}u(x,i)\right|\le\frac{N}{i^2},\quad \sup_{x\in[0,1]}\left|\frac{\mathrm{d}}{\mathrm{d}x}P(x,i)\right|<\infty,\quad i\in\Lambda.$$

Hence the requirements of Definition 6.9 are fulfilled. □

Whatever $t\in[0,1]$ the Markov chain associated with RSCC (6.26) $(s_{n,t})_{n\in\mathbb{N}_+}$ in (4.16) has the transfer operator U in (5.2) with the transition probability function defined as

$$Q(x,B)=\sum_{\{i\in\Lambda:u_{N,i}(x)\in B\}}P_{N,i}(x),\quad x\in[0,1],\ B\in\mathcal{B}_{[0,1]}. \tag{6.27}$$

LEMMA 6.13. *The system* $\left\{([0,1],\mathcal{B}_{[0,1]}),\Lambda,u,P\right\}$ *has a regular associated Markov chain.*

PROOF. By Definition 6.2(v), the Markov chain corresponding to RSCC (6.26) is regular if its transfer operator U is regular with respect to $L([0,1])$. Now, in order to prove the regularity of the associated Markov operator U defined by (5.2) with respect to $L([0,1])$ we have, according to Theorem 6.10(i), to find an element $x^*\in I$ such that $\lim_{n\to\infty}\operatorname{dist}(\sigma_n(x),x^*)=0$ for all $x\in[0,1]$. Here (σ_n) denotes the support of measure $Q^n(w,\cdot)$, where Q^n is the n-step transition probability function of the Markov chain $(s_{n,t})_{n\in\mathbb{N}_+}$.

Fix $x\in[0,1]$. Let us define the sequence $(x_n)_{n\ge0}$ in $[0,1]$, recursively by

$$x_0:=x,\quad x_{n+1}:=1-\frac{N}{x_n+N+1},\quad n\ge1. \tag{6.28}$$

It is clear that $x_n\in(0,1)$, $n\in\mathbb{N}$, and letting $n\to\infty$ in (6.28) we get

$$0<x^*=\lim_{n\to\infty}x_n=\frac{\sqrt{N^2+4}-N}{2}<1 \tag{6.29}$$

Clearly, $x_{n+1}\in\sigma_1(x_n)$ and using Theorem 6.10(ii) and an induction argument lead us to the conclusion that $x_n\in\sigma_n(x)$ for $n\in\mathbb{N}_+$. Since $\operatorname{dist}(\sigma_n(x),x^*)\le|x_n-x^*|\to0$ as $n\to\infty$ we obtain that U is regular. □

PROPOSITION 6.14. $\left\{([0,1],\mathcal{B}_{[0,1]}),\Lambda,u,P\right\}$ *is uniformly ergodic.*

PROOF. We apply Theorem 6.11. Since the RSCC $\left\{([0,1],\mathcal{B}_{[0,1]}),\Lambda,u,P\right\}$ in (6.26) is a RSCC with contraction and is regular then the statement holds. □

Now, Theorem 6.7 implies that $Q^n(\cdot,\cdot)$ converges uniformly to a unique stationary probability measure Q^∞ on $\big([0,1],\mathcal{B}_{[0,1]}\big)$.

PROPOSITION 6.15. *The probability Q^∞ is the invariant probability measure of the transformation R_N.*

PROOF. For ρ_N in (2.6) and Q in (6.27), and on account of the uniqueness of Q^∞ we have to show that

$$\int_0^1 Q(x,B)\,\mathrm{d}\rho_N(x)=\rho_N(B),\quad B\in\mathcal{B}_{[0,1]}. \tag{6.30}$$

Since the intervals $(u,1]\subset[0,1]$ generate $\mathcal{B}_{[0,1]}$, it is sufficient to show the equation (6.30) just for $B=(u,1]$, $0\le u<1$. Now, $u_{N,i}\in B$ iff $i\ge\big\lfloor\frac{N}{1-u}-x\big\rfloor+1$. Thus,

$$\begin{aligned} Q(x,(u,1]) &= \sum_{\{i\in\Lambda:u<u_{N,i}(x)\le 1\}} P_{N,i}(x) \\ &= \sum_{i\ge\lfloor\frac{N}{1-u}-x\rfloor+1} P_{N,i}(x)=\frac{x+N-1}{x+\lfloor\frac{N}{1-u}-x\rfloor+1}. \end{aligned} \tag{6.31}$$

Then,

$$\begin{aligned} \int_0^1 Q(x,(u,1])\,\mathrm{d}\rho_N(x) &= \frac{1}{\log(\frac{N}{N-1})}\int_0^1\frac{\mathrm{d}x}{x+\lfloor\frac{N}{1-u}-x\rfloor} \\ &= \frac{1}{\log(\frac{N}{N-1})}\bigg(\int_0^{\frac{N}{1-u}-\lfloor\frac{N}{1-u}\rfloor}\frac{\mathrm{d}x}{x+\lfloor\frac{N}{1-u}\rfloor}+\int_{\frac{N}{1-u}-\lfloor\frac{N}{1-u}\rfloor}^1\frac{\mathrm{d}x}{x+\lfloor\frac{N}{1-u}-1\rfloor}\bigg) \\ &= \frac{1}{\log(\frac{N}{N-1})}\log\frac{N}{u+n-1}=\rho_N((u,1]). \end{aligned}$$

Hence the statement holds. □

Note that the Markov operator U associated with the RSCC (6.26) is also given by

$$Uf(x)=\int_0^1 f(y)Q(x,\mathrm{d}y) \tag{6.32}$$

with Q as in (6.27), which implies that

$$U^nf(x)=\int_0^1 f(y)Q^n(x,\mathrm{d}y),\quad x\in[0,1],\ n\in\mathbb{N}_+. \tag{6.33}$$

The operator $U^n f$ converges uniformly to a constant function $U^\infty f$ depending on f for any $f \in L([0,1])$ namely

$$U^\infty f = \int_0^1 f(y)\, Q^\infty(\mathrm{d}y). \tag{6.34}$$

Furthermore, there exist two positive constants $q < 1$ and k such that

$$\|U^n f - U^\infty f\|_L \le k q^n \|f\|_L, \quad n \in \mathbb{N}_+,\ f \in L([0,1]). \tag{6.35}$$

Now we are able to show the main result of the paper.

THEOREM 6.16 (A Gauss–Kuzmin-type theorem for R_N). *Fix an integer $N \ge 2$ and let $([0,1], \mathcal{B}_{[0,1]}, R_N)$ be as in Section 2. For a non-atomic probability measure μ which has a Riemann-integrable density on $([0,1], \mathcal{B}_{[0,1]})$, the following holds*:

$$\lim_{n\to\infty} \mu(R_N^n < x) = \frac{1}{\log(\frac{N}{N-1})} \log \frac{x+N-1}{N-1}, \quad x \in [0,1]. \tag{6.36}$$

PROOF. Let ρ_N be as in (2.6). By (5.3), we have

$$\mu\left(R_N^{-n}(A)\right) = \int_A U^n f_0(x)\, \mathrm{d}\rho_N(x) \quad \text{for any } n \in \mathbb{N}_+,\ A \in \mathcal{B}_{[0,1]} \tag{6.37}$$

where $f_0(x) = (x+N-1)(\mathrm{d}\mu/\mathrm{d}\lambda)(x)$ for $x \in [0,1]$. If $\mathrm{d}\mu/\mathrm{d}\lambda \in L([0,1])$, then $f_0 \in L([0,1])$ and by (6.34) and Proposition 6.15 we have

$$U^\infty f_0 = \int_0^1 f_0(x)\, Q^\infty(\mathrm{d}x) = \int_0^1 f_0(x)\, \rho_N(\mathrm{d}x) = \frac{1}{\log(\frac{N}{N-1})}. \tag{6.38}$$

Taking into account (6.35), there exist two constants $q < 1$ and k such that

$$\|U^n f_0 - U^\infty f_0\|_L \le k q^n \|f_0\|_L, \quad n \in \mathbb{N}_+. \tag{6.39}$$

Furthermore, consider the Banach space $C([0,1])$ of all real-valued continuous functions on $[0,1]$ with the norm $\|f\| := \sup_{x\in[0,1]} |f(x)|$. Since $L([0,1])$ is a dense subspace of $C([0,1])$ we have

$$\lim_{n\to\infty} \|(U^n - U^\infty) f\| = 0 \quad \text{for all } f \in C([0,1]). \tag{6.40}$$

Therefore, (6.40) is valid for a measurable function f_0 which is Q^∞-almost surely continuous, that is, for a Riemann-integrable function f. Thus, we have

$$\lim_{n\to\infty} \mu(R_N^n < x) = \lim_{n\to\infty} \int_0^x U^n f_0(u) \rho_N(\mathrm{d}u) = \int_0^x U^\infty f_0(u) \rho_N(\mathrm{d}u)$$

$$= \frac{1}{\log(\frac{N}{N-1})} \int_0^x \rho_N(\mathrm{d}u) = \frac{1}{\log(\frac{N}{N-1})} \log \frac{x+N-1}{N-1}.$$

Hence the proof is complete. □

REMARK 6.17. The estimate of the convergence rate remains an open question. Based on this theoretical framework developed here, in [21] we present an extensive study concerning its optimality using a Wirsing-type approach [23].

Acknowledgement. The authors thank the referee for carefully reading our manuscript and for giving us such constructive comments. These helped us to improving the quality of the paper.

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Publ. Math. Debrecen
96/3-4 (2020), 291–314
DOI: 10.5486/PMD.2020.8536

A two-dimensional Gauss–Kuzmin theorem for N-continued fraction expansions

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Abstract. A two-dimensional Gauss–Kuzmin theorem for N-continued fraction expansions is shown. More precisely, we obtain a Gauss–Kuzmin theorem related to the natural extension of the measure-theoretical dynamical system associated to these expansions. Then, using characteristic properties of the transition operator associated with the random system with complete connections underlying N-continued fractions on the Banach space of complex-valued functions of bounded variation, we derive explicit lower and upper bounds for the convergence rate of the distribution function to its limit.

1. Introduction

The purpose of this paper is to show a two-dimensional Gauss–Kuzmin theorem for N-continued fraction expansions introduced by Burger *et al.* [3], and studied by Dajani *et al.* [6].

In the last twenty-five years, some versions of the so-called *two-dimensional theorem* related to the *Gauss–Kuzmin problem* have been obtained for different types of continued fraction expansions. For example, the case of the regular continued fraction (RCF) expansion was extensively studied in [4], [7] and [9]. For Hurwitz' singular continued fractions one can find the corresponding results in [5] and [14]. In [15], Sebe proved the first two-dimensional Gauss–Kuzmin theorem which leads to an estimate of the approximation error by the expansion algorithm in the grotesque continued fractions.

Mathematics Subject Classification: 11J70, 11K50, 28D05, 60J20.
Key words and phrases: N-continued fractions, Gauss–Kuzmin-problem, natural extension, infinite-order-chain.

Our study is a continuation of the papers [11] and [12], where the second author discussed some metric properties of N-continued fraction expansions. Moreover, he investigated the associated Perron–Frobenius operator and proved a Gauss–Kuzmin theorem for N-continued fractions by applying the method of random systems with complete connections (RSCC) by Iosifescu [8].

1.1. Gauss' Problem and its progress. One of the first and still one of the most important results in the metrical theory of continued fractions is the so-called *Gauss–Kuzmin theorem*. The metrical theory of continued fraction expansions started on 25 October, 1800, with a note by Gauss in his mathematical diary [2]. Any irrational $0 < x < 1$ can be written as the infinite regular continued fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} =: [a_1, a_2, a_3, \ldots], \tag{1.1}$$

where $a_n \in \mathbb{N}_+ := \{1, 2, 3, \ldots\}$ [9]. Such integers $a_1, a_2, \ldots$ are called *incomplete quotients* (or *continued fraction digits*) of x. Define the *regular continued fraction* (or *Gauss*) *transformation* τ on the unit interval $I := [0, 1]$ by

$$\tau(x) = \begin{cases} \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor & \text{if } x \neq 0, \\ \\ 0 & \text{if } x = 0, \end{cases} \tag{1.2}$$

where $\lfloor \cdot \rfloor$ denotes the floor (or entire) function. Denote by $m_n(x)$ the probability that $(\tau^n \leq x)$. Although he does not explicitly state it, Gauss must have known the recursion formula

$$m_{n+1}(x) = \sum_{i=1}^{\infty} \left(m_n\left(\frac{1}{i}\right) - m_n\left(\frac{1}{x+i}\right)\right), \tag{1.3}$$

since he wrote that he could prove by a "very simple reasoning" that (in modern notation)

$$\lim_{n\to\infty} m_n(x) = \frac{\log(1+x)}{\log 2}, \quad x \in I. \tag{1.4}$$

In 1812, Gauss asked Laplace [2] to estimate the *n-th error term* $e_n(x)$ defined by

$$e_n(x) := m_n(x) - \frac{\log(1+x)}{\log 2}, \quad n \geq 1,\ x \in I. \tag{1.5}$$

This has been called *Gauss' Problem*. It received a first solution more than a century later, when KUZMIN [10] showed in 1928 that $e_n(x) = \mathcal{O}(q^{\sqrt{n}})$ as $n \to \infty$, uniformly in x with some (unspecified) $0 < q < 1$. This has been called the *Gauss–Kuzmin theorem* and the recursion formula (1.3) is the *Gauss–Kuzmin equation*.

For such historical reasons, the *Gauss–Kuzmin theorem* is regarded as the first basic result in the rich metrical theory of continued fractions. By such a development, generalizations of these problems for non-regular continued fractions are also called as the *Gauss–Kuzmin problem*.

1.2. Outline. In Section 2, we present notions and preliminary results to be used in the sequel. We mention that the infinite-order-chain representation of the sequence of the partial quotients of the N-continued fraction expansion allows a concise formulation of the obtained results. Section 3 presents a Gauss–Kuzmin theorem related to the natural extension [13] of the measure-theoretical dynamical system associated to N-continued fraction expansions. Section 4 is devoted to deriving explicit lower and upper bounds of the error which provide a more refined estimate of the convergence rate. The key role in this section is played by the transition operator associated with the RSCC underlying N-continued fraction on the Banach space of functions of bounded variation. The last section collects some concluding remarks and gives some numerical calculations.

2. Preliminaries

Fix an integer $N \geq 1$. The measure-theoretical dynamical system $(I, \mathcal{B}_I, T_N)$ is defined as follows: $I := [0, 1]$, $\mathcal{B}_I$ denotes the σ-algebra of all Borel subsets of I, and T_N is the transformation

$$T_N : I \to I; \quad T_N(x) := \begin{cases} \dfrac{N}{x} - \left\lfloor \dfrac{N}{x} \right\rfloor & \text{if } x \in (0, 1], \\ 0 & \text{if } x = 0. \end{cases} \tag{2.1}$$

In addition, we write $(I, \mathcal{B}_I, G_N, T_N)$ as $(I, \mathcal{B}_I, T_N)$ with the following probability measure G_N on $(I, \mathcal{B}_I)$:

$$G_N(A) := \frac{1}{\log \frac{N+1}{N}} \int_A \frac{\mathrm{d}x}{x + N}, \quad A \in \mathcal{B}_I. \tag{2.2}$$

The measure G_N is T_N-invariant, i.e., $G_N\left(T_N^{-1}(A)\right) = G_N(A)$ for any $A \in \mathcal{B}_I$, and the dynamical system $(I, \mathcal{B}_I, G_N, T_N)$ is ergodic [6]. For any $0 < x < 1$,

put $a_1(x) = \lfloor N/x \rfloor$ and $a_n(x) = a_1\left(T_N^{n-1}(x)\right)$, $n \in \mathbb{N}_+$, with $T_N^0(x) = x$. Then every irrational $0 < x < 1$ can be written in the form

$$x = \cfrac{N}{a_1 + \cfrac{N}{a_2 + \cfrac{N}{a_3 + \ddots}}} =: [a_1, a_2, a_3, \ldots]_N, \tag{2.3}$$

where a_n's are non-negative integers, $a_n \geq N$, $n \in \mathbb{N}_+ := \{1, 2, \ldots\}$. We call (2.3) the *$N$-continued fraction expansion* of x and $p_n(x)/q_n(x) := [a_1, a_2, a_3, \ldots, a_n]_N$ the *n-th order convergent* of $x \in I$. Then $p_n(x)/q_n(x) \to x$, $n \to \infty$. Here p_n's and q_n's satisfy for $n \in \mathbb{N}_+$ the following:

$$p_n(x) := a_n p_{n-1} + N p_{n-2}, \quad n \geq 1, \tag{2.4}$$

$$q_n(x) := a_n q_{n-1} + N q_{n-2}, \quad n \geq 1, \tag{2.5}$$

with $p_0(x) := 0$, $q_0(x) := 1$, $p_{-1}(x) := 1$, $q_{-1}(x) := 0$, $p_1(x) := N$, $q_1(x) := a_1$. In [12], we showed that

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| < \frac{N^n}{q_n^2(x)}, \quad n \geq 1. \tag{2.6}$$

For any $n \in \mathbb{N}_+$ and $i^{(n)} = (i_1, \ldots, i_n) \in \Lambda^n$, $\Lambda := \{N, N+1, \ldots\}$, we call

$$I_N\left(i^{(n)}\right) = \{x \in I \setminus \mathbb{Q} : a_k(x) = i_k \text{ for } k = 1, \ldots, n\} \tag{2.7}$$

a *fundamental interval of rank n* and make the convention that $I_N(i^{(0)}) = I \setminus \mathbb{Q}$. For example, for any $i \in \Lambda$, we have

$$I_N(i) = \{x \in I \setminus \mathbb{Q} : a_1 = i\} = (I \setminus \mathbb{Q}) \cap \left(\frac{N}{i+1}, \frac{N}{i} \right). \tag{2.8}$$

Denote by λ the Lebesgue measure on I. For any $n \in \mathbb{N}_+$, the conditional probability $\lambda(T_N^n < x | a_1, \ldots, a_n)$ is given as follows [12]:

$$\lambda(T_N^n < x | a_1, \ldots, a_n) = \frac{(s_n + N)x}{s_n x + N}, \quad x \in I, \tag{2.9}$$

where $s_n := N\frac{q_{n-1}}{q_n} = [a_n, a_{n-1}, \ldots, a_2, a_1]_N$, $n \geq 1$, $s_0 := 0$. Equation (2.9) is the Brodén–Borel–Lévy formula for this type of expansion. It allows us to determine the probability distribution of $(a_n)_{n \in \mathbb{N}_+}$ under λ. For any $i \in \Lambda$, we have [12]:

$$\lambda(a_1 = i) = \frac{N}{i(i+1)}, \quad \lambda\left(a_{n+1} = i | a_1, \ldots, a_n\right) = V_{N,i}(s_n), \tag{2.10}$$

where

$$V_{N,i}(x) := \frac{x+N}{(x+i)\,(x+i+1)}. \tag{2.11}$$

Hence the invariant probability measure of the transformation T_N is G_N, the sequence $(a_n)_{n\in\mathbb{N}_+}$ is strictly stationary on $(I, \mathcal{B}_I, G_N)$. As such, a doubly infinite version of it, say $(\overline{a}_l)_{l\in\mathbb{Z}}$, should exist on a richer probability space. Indeed, such a version can be effectively constructed on the natural extension $(I^2, \mathcal{B}_{I^2}, \overline{G_N}, \overline{T_N})$ of $(I, \mathcal{B}_I, G_N, T_N)$, where the transformation $\overline{T_N}$ is defined on the product space $(I^2, \mathcal{B}_I^2) := (I, \mathcal{B}_I) \times (I, \mathcal{B}_I)$ by

$$\overline{T_N}(x,y) := \left(T_N(x),\, \frac{N}{a_1(x)+y}\right), \tag{2.12}$$

and the extended measure $\overline{G_N}$ is defined by

$$\overline{G_N}(B) := \frac{1}{\log\left(\frac{N+1}{N}\right)} \iint_B \frac{N\mathrm{d}x\mathrm{d}y}{(xy+N)^2}, \quad B \in \mathcal{B}_I^2. \tag{2.13}$$

In [6], it was shown that the measure $\overline{G_N}$ is preserved by $\overline{T_N}$, that is, $\overline{G_N}(B) = \overline{G_N}\left((\overline{T_N})^{-1}(B)\right)$, for any $B \in \mathcal{B}_I^2$. The transformation $\overline{T_N}$ is bijective on I^2 with the inverse

$$(\overline{T_N})^{-1}(x,y) = \left(\frac{N}{a_1(y)+x},\, T_N(y)\right), \quad (x,y) \in I^2. \tag{2.14}$$

Iterations of (2.12) and (2.14) are given as follows for each $n \geq 2$:

$$\begin{aligned}
\left(\overline{T_N}\right)^n (x,y) &= \left(T_N^n(x), [a_n(x), a_{n-1}(x), \ldots, a_2(x), a_1(x)+y]_N\right), \\
\left(\overline{T_N}\right)^{-n} (x,y) &= \left([a_n(y), a_{n-1}(y), \ldots, a_2(y), a_1(y)+x]_N, T_N^n(y)\right).
\end{aligned}$$

We define the *extended incomplete quotients* $\overline{a}_l(x,y)$, $l \in \mathbb{Z}$ at $(x,y) \in I^2$ by

$$\overline{a}_l(x,y) := \overline{a}_1\left((\overline{T_N})^{l-1}(x,y)\right), \quad l \in \mathbb{Z}, \tag{2.15}$$

with $\overline{a}_1(x,y) = a_1(x)$. Clearly, for $n \in \mathbb{N}_+$ and any $(x,y) \in I^2$, we have $\overline{a}_n(x,y) = a_n(x)$, $\overline{a}_0(x,y) = a_1(y)$, $\overline{a}_{-n}(x,y) = a_{n+1}(y)$. Since the extended measure $\overline{G_N}$ is preserved by $\overline{T_N}$, the doubly infinite sequence $(\overline{a}_l(x,y))_{l\in\mathbb{Z}}$ is strictly stationary under $\overline{G_N}$. In [12], we showed that the dependence structure of $(\overline{a}_l)_{l\in\mathbb{Z}}$ follows from the fact that

$$\overline{G_N}([0,x] \times I \,|\, \overline{a}_0, \overline{a}_{-1}, \ldots) = \frac{(N+a)x}{ax+N}, \quad \overline{G_N}\text{-a.s.} \tag{2.16}$$

where $a := [\overline{a}_0, \overline{a}_{-1}, \ldots]_N$. Hence, we obtained in [12] that

$$\overline{G_N}(\overline{a}_1 = i | \, \overline{a}_0, \overline{a}_{-1}, \ldots) = V_{N,i}(a), \quad \overline{G_N}\text{-a.s.} \tag{2.17}$$

where $V_{N,i}$ is as in (2.11). We thus see that $(\overline{a}_l)_{l\in\mathbb{Z}}$ is an infinite-order-chain in the theory of dependence with complete connections [8].

Put $\overline{s}_\ell = [\overline{a}_\ell, \overline{a}_{\ell-1}, \ldots]_N$, $\ell \in \mathbb{Z}$. Note that $\overline{s}_\ell = N/(\overline{s}_{\ell-1} + \overline{a}_\ell)$, $\ell \in \mathbb{Z}$. It follows from (2.17) that $(\overline{s}_\ell)_{\ell\in\mathbb{Z}}$ is a strictly stationary I-valued Markov process on $(I^2, \mathcal{B}_I^2, \overline{G_N})$ with the following transition mechanism: from state $\overline{s} \in I$ the only possible transitions are those to states $N/(\overline{s}+i)$, $i \in \Lambda$, the transition probability being $V_{N,i}(\overline{s})$. Clearly, whatever $\ell \in \mathbb{Z}$ we have

$$\overline{G_N}\,(\overline{s}_\ell < x) = \overline{G_N}\,(\overline{s}_0 < x) = \overline{G_N}(I \times [0, x]) = G_N([0, x]), \quad x \in I.$$

Motivated by (2.16), we shall consider the family of (conditional) probability measures $(G_{N,t})_t$ on $\mathcal{B}_I$ defined by their distribution functions

$$G_{N,t}([0, x]) = \frac{(N + t)x}{tx + N}, \quad x, t \in I. \tag{2.18}$$

In particular, $G_{N,0}$ is the Lebesgue measure on I. The density $h_{N,t}$ of $G_{N,t}$ is

$$h_{N,t}(x) = \frac{N(N + t)}{(tx + N)^2}, \quad x \in I, \ t \in I.$$

Using a similar reasoning as for regular continued fractions [9, Section 1.3.4] and [1, p. 224], we get

$$\begin{aligned}\sup_{A\in\mathcal{B}_I} |G_{N,t}(A) - G_{N,r}(A)| &= \frac{1}{2}\int_I |h_{N,t}(x) - h_{N,r}(x)| \, \mathrm{d}x \\ &= \frac{N}{2}|r - t| \int_I \frac{\left|(N(r + t) + rt)x^2 + 2xN^2 - N^2\right|}{(tx + N)^2(rx + N)^2}\mathrm{d}x = k_N|r - t|, \end{aligned} \tag{2.19}$$

where

$$k_N = \frac{N}{2}\int_I \frac{\left|(N(r + t) + rt)x^2 + 2xN^2 - N^2\right|}{(tx + N)^2(rx + N)^2}\mathrm{d}x, \quad r, t \in I.$$

For any $t \in I$, put $s_{0,t} = t$ and $s_{n,t} = N/\left(s_{n-1,t} + a_n\right)$, $n \in \mathbb{N}_+$. The motivation for considering the $s_{n,t}$, $n \in \mathbb{N}$, $t \in I$, will appear later on, see (2.25). Note that by the very definition of $s_{n,t}$, we have $s_{n,t} = [a_n, \ldots, a_2, a_1 + t]_N$, $n \geq 2$, while $s_{1,t} = N/(a_1 + t)$, $t \in I$. These facts lead us to the random system with

complete connections [11] $\{(I,\mathcal{B}_I),(\Lambda,\mathcal{P}(\Lambda)),u,V\}$, where $\mathcal{P}(\Lambda)$ is the power set of Λ, $u: I\times\Lambda\to I$ is defined as

$$u(x,i):=u_i(x)=u_{N,i}(x)=\frac{N}{x+i}, \tag{2.20}$$

and $V: I\times\Lambda\to I$ is defined as

$$V(x,i):=v_i(x)=V_{N,i}(x) \tag{2.21}$$

with $V_{N,i}$ as in (2.11), for all $x\in I$ and $i\in\Lambda$.

Then $(s_{n,t})_{n\in\mathbb{N}_+}$ is an I-valued Markov chain on $(I,\mathcal{B}_I,G_{N,t})$ which starts from $s_{t,0}=t$, $t\in I$, and has the following transition mechanism: from state $s\in I$ the only possible transitions are those to states $N/(s+i)$ with the corresponding transition probability $V_{N,i}(s)$, $i\in\Lambda$. Let $B(I)$ denote the Banach space of all bounded $\mathcal{B}_I$-measurable complex-valued functions defined on I, which is a Banach space under the supremum norm. The transition operator U of $(s_{n,t})_{n\in\mathbb{N}_+}$ sends $f\in B(I)$ into the function defined by

$$E_t\left(f(s_{n+1,t})|\,s_{n,t}=s\right)=\sum_{i\in\Lambda}V_{N,i}(s)f\left(\frac{N}{s+i}\right)=:Uf(s) \tag{2.22}$$

for any $s\in I$, where E_t stands for the mean-value operator with respect to the probability measure $G_{N,t}$, whatever $t\in I$, and U is the Perron–Frobenius operator of $(I,\mathcal{B}_I,G_N,T_N)$ [12].

In connection with the operator U, if we define

$$U^\infty f=\int_I f(x)\mathrm{d}G_N(x),\quad f\in B(I), \tag{2.23}$$

then we have

$$U^\infty U^n f=U^\infty f,\qquad \text{for any } f\in B(I) \text{ and } n\in\mathbb{N}_+. \tag{2.24}$$

Note that for any $t\in I$ and $n\in\mathbb{N}_+$, we have

$$G_{N,t}\left(A|a_1,\dots,a_n\right)=G_{N,s_{n,t}}\left(T_N^n(A)\right),$$

whatever the set A belonging to the σ-algebra generated by the random variables $a_{n+1},a_{n+2}\dots$, that is, $\sigma(a_{n+1},a_{n+2},\dots)=T_N^{-n}(\mathcal{B}_I)$. This follows from (2.16) for all irrational $t\in I$ and by continuity (use (2.19)) for all rational $t\in I$. In particular, it follows that the Brodén–Borel–Lévy formula holds under $G_{N,t}$ for any $t\in I$, that is,

$$G_{N,t}(T_N^n<x|a_1,\dots,a_n)=\frac{(s_{n,t}+N)x}{s_{n,t}x+N},\quad x\in I,\ n\in\mathbb{N}_+. \tag{2.25}$$

Note also that $(s_{n,t})_{n\in\mathbb{N}}$ under $G_{N,t}$ is a version of $(\overline{s}_n)_{n\in\mathbb{N}}$ under $\overline{G_N}(\cdot|\overline{s}_0=t)$ for any $t\in I$.

3. A two-dimensional Gauss–Kuzmin theorem

In this section, we give a Gauss–Kuzmin theorem related to the natural extension $\left(I^2, \mathcal{B}_I^2, \overline{G_N}, \overline{T_N}\right)$. To solve the problem, we need a version of the Gauss–Kuzmin theorem for T_N different from that presented by the second author in [11]. The problem of finding the asymptotic behaviour of $T_N^{-n}(A)$ as $n \to \infty$, $A \in \mathcal{B}_I$, represents the Gauss–Kuzmin problem for N-continued fraction expansions.

Theorem 3.1 (a Gauss–Kuzmin theorem for T_N)**.** *Let (I, G_N, T_N) as in* (2.2) *and* (2.1)*. Denote by λ the Lebesgue measure on I. Then there exists a constant $0 < \theta < 1$ such that for any $A \in \mathcal{B}_I$, we have*

$$\left|\lambda\left(T_N^{-n}(A)\right) - G_N(A)\right| < C\theta^n \lambda(A), \tag{3.1}$$

where C is a universal constant.

Proof. In [12, (Proposition 8.1(ii)(c))], we showed that $\mu\left((T_N)^{-n}(A)\right) = \int_A U^n f(x) \mathrm{d}G_N(x)$, where μ is a probability measure on $(I, \mathcal{B}_I)$ absolutely continuous with respect to the Lebesgue measure λ, and $f(x) := \left(\log\left(\frac{N+1}{N}\right)\right)(x+N)h(x)$ with $h := d\mu/d\lambda$ a.e. in I. In the special case $\mu = \lambda$, we obviously have

$$\lambda\left(T_N^{-n}(A)\right) = \frac{1}{\log\left(\frac{N+1}{N}\right)} \int_A \frac{U^n f(x)}{x+N} \mathrm{d}x \tag{3.2}$$

with $f(x) := \left(\log\left(\frac{N+1}{N}\right)\right)(x+N)$, $x \in I$. Thus, from (2.23) we have that $U^\infty f = 1$. Therefore,

$$G_N(A) = \frac{1}{\log\left(\frac{N+1}{N}\right)} \int_A \frac{U^\infty f(x)}{x+N} \mathrm{d}x. \tag{3.3}$$

In [11], we showed that there are two positive constants $\theta < 1$ and k such that

$$\left\|U^n g - U^\infty g\right\|_L \leq k\theta^n \left\|g\right\|_L, \quad g \in L(I),\ n \in \mathbb{N}_+, \tag{3.4}$$

where $L(I)$ denotes the Banach space of all complex-valued Lipschitz continuous functions on I with the following norm:

$$\left\|g\right\|_L := \sup_{x \in I} |g(x)| + \sup_{x' \neq x''} \frac{|g(x') - g(x'')|}{|x' - x''|}.$$

Therefore

$$\begin{aligned}\left|\lambda\left(T_N^{-n}(A)\right) - G_N(A)\right| &\leq \frac{1}{\log\left(\frac{N+1}{N}\right)} \int_A \frac{|U^n f(x) - U^\infty f(x)|}{x+N} \mathrm{d}x \\ &< k\theta^n \left\|f\right\|_L \frac{1}{\log\left(\frac{N+1}{N}\right)} \int_A \frac{1}{x+N} \mathrm{d}x = k\theta^n \left\|f\right\|_L \theta^n G_N(A).\end{aligned}$$

Since

$$G_N(A) \le \frac{1}{N \log\left(\frac{N+1}{N}\right)} \lambda(A), \quad A \in \mathcal{B}_I,$$

the proof is completed. □

The essential argument in the proof of the Gauss–Kuzmin theorem for $\overline{T_N}$ is the Gauss–Kuzmin equation. First we give the Gauss–Kuzmin equation for the one-dimensional case. Define the functions $(F_n)_{n\in\mathbb{N}}$ on I by

$$F_0(x) := x, \quad F_n(x) := \lambda(T_N^n < x), \quad n \ge 1. \tag{3.5}$$

The Gauss–Kuzmin equation is

$$F_{n+1}(x) = \sum_{i\in\Lambda} \left(F_n\left(\frac{N}{i}\right) - F_n\left(\frac{N}{x+i}\right)\right) \tag{3.6}$$

for $x \in I$ and $n \in \mathbb{N}$. The density of the measure G_N defined in (2.2) is an eigenfunction of (3.6), namely, if we put $F_n(x) = \log\left(\frac{x+N}{N}\right)$, $x \in I$, we obtain $F_{n+1}(x) = \log\left(\frac{x+N}{N}\right)$.

Next we will derive a Gauss–Kuzmin theorem related to the natural extension $(I^2, \mathcal{B}_I^2, \overline{G_N}, \overline{T_N})$ defined in Section 2.

For any $n \in \mathbb{N}_+$ and $x, y \in I$, let us define $\Delta_{x,y} := [0,x] \times [0,y]$ and the functions $\overline{F}_n(x,y)$ by

$$\overline{F}_n(x,y) := \overline{\lambda}\left(\left(\overline{T_N}\right)^n \in \Delta_{x,y}\right), \tag{3.7}$$

where $\overline{\lambda}$ is the Lebesgue measure on I^2. Then the following holds.

Theorem 3.2 (a Gauss–Kuzmin theorem for $\overline{T_N}$)**.** *For every $n \ge 2$ and $(x,y) \in I^2$, one has*

$$\overline{F}_n(x,y) = \overline{G_N}\left(\Delta_{x,y}\right) = \frac{1}{\log\left(\frac{N+1}{N}\right)} \log\left(\frac{xy+N}{N}\right) + \mathcal{O}\left(\rho^n\right) \tag{3.8}$$

with $0 < \rho < 1$.

For any $0 < y \le 1$, put $\ell_1 := \left\lfloor \frac{N}{y} \right\rfloor$. For $(\xi,\zeta) \in I^2$, then $\left(\overline{T_N}\right)^{n+1}(\xi,\zeta) \in \Delta_{x,y}$ is equivalent to

$$\left(\overline{T_N}\right)^n \in \left(\bigcup_{i \ge \ell_1+1} \left[\frac{N}{x+i}, \frac{N}{i}\right] \times [0,1] \right) \cup \left(\left[\frac{N}{x+\ell_1}, \frac{N}{\ell_1}\right] \times \left[\frac{N}{y} - \ell_1, 1\right] \right).$$

Now, from (3.7) we have the following recursive formula that represents the Gauss–Kuzmin equation associated to the functions $(\overline{F}_n)_{n\in\mathbb{N}_+}$:

$$\begin{aligned}\overline{F}_{n+1}(x,y) = \sum_{i\geq \ell_1}\left(\overline{F}_n\left(\frac{N}{i},1\right) - \overline{F}_n\left(\frac{N}{x+i},1\right)\right)\\ -\left(\overline{F}_n\left(\frac{N}{\ell_1},\frac{N}{y}-\ell_1\right) - \overline{F}_n\left(\frac{N}{x+\ell_1},\frac{N}{y}-\ell_1\right)\right).\end{aligned} \tag{3.9}$$

The density of the measure $\overline{G_N}$ defined in (2.13) is an eigenfunction of (3.9), namely, if we put $\overline{F}_n(x,y) = \log\left(\frac{xy+N}{N}\right)$, $x,y\in I$, we obtain $\overline{F}_{n+1}(x,y) = \log\left(\frac{xy+N}{N}\right)$.

Lemma 3.1. *Let $n\in\mathbb{N}$, $n\geq 2$, and let $y\in I\cap\mathbb{Q}$ with $y=[\ell_1,\dots,\ell_d]_N$, $\ell_1,\dots,\ell_d\in\Lambda$, $\ell_d\geq N+1$, where $d\leq \lfloor n/(N+1)\rfloor$. Then for every $x,x^*\in[0,1)$ with $x^*<x$,*

$$\left|\overline{F}_n(x,y) - \overline{F}_n(x^*,y) - \frac{\log\left(\frac{xy+N}{x^*y+N}\right)}{\log\left(\frac{N+1}{N}\right)}\right| < \frac{(N+1)^3}{N^2}\frac{C}{1-q}\overline{\lambda}(\Delta_{x,y}\setminus\Delta_{x^*,y})q^{n-d},$$

where $0<q<1$, and C is given as in Theorem 3.1.

Proof. Let $y_0 = y$, $y_i := [\ell_{i+1},\dots,\ell_d]_N$, $i=1,\dots,d-1$, and $y_d = 0$. Applying (3.9) one gets

$$\begin{aligned}\overline{F}_n(x,y) - \overline{F}_n(x^*,y) = \sum_{i\geq\ell_1}\left(\overline{F}_{n-1}\left(\frac{N}{x^*+i},1\right) - \overline{F}_{n-1}\left(\frac{N}{x+i},1\right)\right)\\ +\left(\overline{F}_{n-1}\left(\frac{N}{x+\ell_1},y_1\right) - \overline{F}_{n-1}\left(\frac{N}{x^*+\ell_1},y_1\right)\right).\end{aligned} \tag{3.10}$$

Now for each $B\in\mathcal{B}_I^2$ one has

$$\frac{N}{(N+1)^2\log\left(\frac{N+1}{N}\right)}\overline{\lambda}(B) \leq \overline{G_N}(B) \leq \frac{1}{N\log\left(\frac{N+1}{N}\right)}\overline{\lambda}(B). \tag{3.11}$$

Now from (2.8), (3.11) and the fact that $\overline{T_N}$ is $\overline{G_N}$-invariant, it follows that

$$\begin{aligned}&\sum_{i\geq\ell_1}\lambda\left(\left[\frac{N}{x+i},\frac{N}{x^*+i}\right]\right)\\ &\quad=\sum_{i\geq\ell_1}\left(\frac{N}{x^*+i}-\frac{N}{x+i}\right) = \sum_{i\geq\ell_1}\overline{\lambda}\left(\left[\frac{N}{x+i},\frac{N}{x^*+i}\right]\times[0,1]\right)\end{aligned}$$

$$\leq \frac{(N+1)^2}{N}\log\left(\frac{N+1}{N}\right)\sum_{i\geq\ell_1}\overline{G_N}\left((x^*,x)\times I_N(i)\right)$$
$$\leq \frac{(N+1)^2}{N^2}\sum_{i\geq\ell_1}\overline{\lambda}\left((x^*,x)\times I_N(i)\right) = \frac{(N+1)^2}{N^2}(x-x^*)\sum_{i\geq\ell_1}\lambda\left(I_N(i)\right)$$
$$= \frac{(N+1)^2}{N^2}(x-x^*)\sum_{i\geq\ell_1}\left(\frac{N}{i}-\frac{N}{i+1}\right) \leq \frac{(N+1)^2}{N^2}(x-x^*)\frac{N}{\ell_1}$$
$$\leq \frac{(N+1)^3}{N^2}(x-x^*)y = \frac{(N+1)^3}{N^2}\overline{\lambda}\left(\Delta_{x,y}\setminus\Delta_{x^*,y}\right). \tag{3.12}$$

Since for every $2\leq k\leq d$, $\lambda\left(I_N\left(\ell_1,\ldots,\ell_{k-1}\right)\right)\leq (N+1)y$, a similar analysis leads to

$$\sum_{i\geq\ell_k}\left|[i,\ell_{k-1},\ldots,x^*+\ell_1]_N-[i,\ell_{k-1},\ldots,x+\ell_1]_N\right| \leq \frac{(N+1)^2}{N}\log\left(\frac{N+1}{N}\right)$$
$$\times\sum_{i\geq\ell_k}\overline{G_N}\left(\left(\overline{T_N}\right)^k\left([i,\ell_{k-1},\ldots,x^*+\ell_1]_N,[i,\ell_{k-1},\ldots,x+\ell_1]_N\right)\times[0,1]\right)$$
$$\leq \frac{(N+1)^2}{N^2}\sum_{i\geq\ell_k}\overline{\lambda}\left((x^*,x)\times I_N\left(\ell_1,\ldots,\ell_{k-1},i\right)\right)$$
$$\leq \frac{(N+1)^2}{N^2}(x-x^*)\lambda\left(I_N\left(\ell_1,\ldots,\ell_{k-1}\right)\right)$$
$$\leq \frac{(N+1)^3}{N^2}(x-x^*)y = \frac{(N+1)^3}{N^2}\overline{\lambda}\left(\Delta_{x,y}\setminus\Delta_{x^*,y}\right). \tag{3.13}$$

Since $\overline{F}_n(x,1)=F_n(x)$, from Theorem 3.1 it follows that

$$\sum_{i\geq\ell_1}\left(\overline{F}_{n-1}\left(\frac{N}{x^*+i},1\right)-\overline{F}_{n-1}\left(\frac{N}{x+i},1\right)\right)$$
$$=\sum_{i\geq\ell_1}\left(G_N\left(\left[\frac{N}{x+i},\frac{N}{x^*+i}\right]\right)+\left(\frac{N}{x^*+i}-\frac{N}{x+i}\right)\mathcal{O}\left(q^{n-1}\right)\right)$$
$$=\frac{1}{\log\left(\frac{N+1}{N}\right)}\sum_{i\geq\ell_1}\log\left(\frac{x^*+i+1}{x+i+1}\cdot\frac{x+i}{x^*+i}\right)+\frac{(N+1)^3}{N^2}\overline{\lambda}\left(\Delta_{x,y}\setminus\Delta_{x^*,y}\right)\mathcal{O}\left(q^{n-1}\right)$$
$$=\frac{1}{\log\left(\frac{N+1}{N}\right)}\cdot\log\left(\frac{x+\ell_1}{x^*+\ell_1}\right)+\frac{(N+1)^3}{N^2}\overline{\lambda}\left(\Delta_{x,y}\setminus\Delta_{x^*,y}\right)\mathcal{O}\left(q^{n-1}\right). \tag{3.14}$$

Now by (3.10) and (3.14), we have

$$\begin{aligned}
&\overline{F}_n(x,y)-\overline{F}_n(x^*,y)\\
&\quad=\frac{1}{\log\left(\frac{N+1}{N}\right)}\log\left(\frac{x+\ell_1}{x^*+\ell_1}\right)+\frac{(N+1)^3}{N^2}\overline{\lambda}\left(\Delta_{x,y}\setminus\Delta_{x^*,y}\right)\mathcal{O}\left(q^{n-1}\right)\\
&\qquad+\overline{F}_{n-1}\left(\frac{N}{\ell_1+x},y_1\right)-\overline{F}_{n-1}\left(\frac{N}{\ell_1+x^*},y_1\right).
\end{aligned}$$

Applying (3.10) and Theorem 3.1 again, we have

$$\begin{aligned}
&\overline{F}_{n-1}\left(\frac{N}{\ell_1+x},y_1\right)-\overline{F}_{n-1}\left(\frac{N}{\ell_1+x^*},y_1\right)\\
&\quad=\frac{1}{\log\left(\frac{N+1}{N}\right)}\log\left(\frac{\ell_2+\frac{N}{x+\ell_1}}{\ell_2+\frac{N}{x^*+\ell_1}}\right)+\frac{(N+1)^3}{N^2}\overline{\lambda}\left(\Delta_{x,y}\setminus\Delta_{x^*,y}\right)\mathcal{O}\left(q^{n-2}\right)\\
&\qquad+\overline{F}_{n-2}\left(\frac{N}{\ell_2+\frac{N}{\ell_1+x}},y_2\right)-\overline{F}_{n-2}\left(\frac{N}{\ell_2+\frac{N}{\ell_1+x^*}},y_2\right).
\end{aligned}$$

Applying (3.10) d-times and taking into account that $y_d=0$, we get

$$\begin{aligned}
&\overline{F}_n(x,y)-\overline{F}_n(x^*,y)\\
&\quad=\frac{1}{\log\left(\frac{N+1}{N}\right)}\log\left(\frac{x+\ell_1}{x^*+\ell_1}\cdot\frac{[x+\ell_1]_N+\ell_2}{[x^*+\ell_1]_N+\ell_2}\cdot\frac{[\ell_{d-1},\ldots,\ell_2,x+\ell_1]_N+\ell_d}{[\ell_{d-1},\ldots,\ell_2,x^*+\ell_1]_N+\ell_d}\right)\\
&\qquad+\frac{(N+1)^3}{N^2}\overline{\lambda}\left(\Delta_{x,y}\setminus\Delta_{x^*,y}\right)\left(\mathcal{O}\left(q^{n-1}\right)+\ldots+\mathcal{O}\left(q^{n-d}\right)\right).
\end{aligned}\tag{3.15}$$

If p_d and q_d are as in (2.4) and (2.5) with $a_1=x+\ell_1$ and $a_d=\ell_d$, $d\geq 2$, then $[\ell_i,\ldots,\ell_2,x+\ell_1]_N=N\dfrac{q_{i-1}}{q_i}$, $i=1,\ldots,d$. Since $q_0=1$, we have

$$(x+\ell_1)([x+\ell_1]_N+\ell_2)\cdots([\ell_{d-1},\ldots,\ell_2,x+\ell_1]_N+\ell_d)=q_d.$$

Let p_d^* and q_d^* be as in (2.4) and (2.5), with $a_1=x^*+\ell_1$ and $a_d=\ell_d$, $d\geq 2$. Since p_d and p_d^* do not depend on the a_1, we have that $p_d=p_d^*$. Thus we find that

$$\begin{aligned}
&\frac{(x+\ell_1)([x+\ell_1]_N+\ell_2)\cdots([\ell_{d-1},\ldots,\ell_2,x+\ell_1]_N+\ell_d)}{(x^*+\ell_1)([x^*+\ell_1]_N+\ell_2)\cdots([\ell_{d-1},\ldots,\ell_2,x^*+\ell_1]_N+\ell_d)}\\
&\qquad=\frac{q_d}{q_d^*}=\frac{p_d^*}{q_d^*}\frac{q_d}{p_d}=\frac{x+\ell_1+[\ell_2,\ldots,\ell_d]_N}{x^*+\ell_1+[\ell_2,\ldots,\ell_d]_N}=\frac{x+\frac{N}{y}}{x^*+\frac{N}{y}}=\frac{xy+N}{x^*y+N}.
\end{aligned}\tag{3.16}$$

Since $q^{n-d}+\cdots+q^{n-1}=q^{n-d}\dfrac{1-q^d}{1-q}<\dfrac{q^{n-d}}{1-q}$, from (3.15) and (3.16) we obtain the desired result. □

PROOF OF THEOREM 3.2. Let $(x,y) \in I^2$, $n \geq 2$. In view of Lemma 3.1, we assume that $y \notin \mathbb{Q}$. Put $d := \max\{\kappa \in \mathbb{N} : \kappa \text{ even and } \kappa \leq \lfloor n/(N+1) \rfloor + 2\}$.

Let q_n be as in (2.5). Since $a_n \geq N$, $n \in \mathbb{N}_+$, we have $q_n \geq N(q_{n-1} + q_{n-2})$, $n \in \mathbb{N}_+$. Define now $\overline{q}_n$, $n \in \mathbb{N}_+$, by the recurrence relation

$$\overline{q}_n = N(\overline{q}_{n-1} + \overline{q}_{n-2}), \quad n \in \mathbb{N}_+, \tag{3.17}$$

with $\overline{q}_{-1} = 0$ and $\overline{q}_0 = 1$. Then

$$q_n \geq \overline{q}_n, n \in \mathbb{N}_+. \tag{3.18}$$

It is easy to see that

$$\overline{q}_n = \frac{1}{\sqrt{N^2+4N}} \left[\left(\frac{N+\sqrt{N^2+4N}}{2} \right)^{n+1} - \left(\frac{N-\sqrt{N^2+4N}}{2} \right)^{n+1} \right]. \tag{3.19}$$

From (2.6), (3.18), (3.19) and the fact that d is even and $\frac{n}{N+1} < d \leq \frac{n}{N+1} + 2$, we get

$$\begin{aligned} \left| y - \frac{p_d}{q_d} \right| &< \frac{N^d}{(\overline{q}_d)^2} = \frac{N^d \left(N^2+4N\right)}{\left[\left(\frac{N+\sqrt{N^2+4N}}{2} \right)^{d+1} - \left(\frac{N-\sqrt{N^2+4N}}{2} \right)^{d+1} \right]^2} \\ &< \frac{N^d \left(N^2+4N\right)}{\left[\left(\frac{N+\sqrt{N^2+4N}}{2} \right)^{d+1} \right]^2} < \frac{(N)^{\frac{n}{N+1}+2} \cdot (N^2+4N) \cdot 2^{\frac{2n}{N+1}+2}}{\left(N+\sqrt{N^2+4N}\right)^{\frac{2n}{N+1}+2}}. \end{aligned} \tag{3.20}$$

Since $\Delta_{x,p_d/q_d} \subset \Delta_{x,y}$ and $\overline{F}_n(x,y) = \overline{\lambda}\left(\left(\overline{T_N}\right)^{-n} (\Delta_{x,y}) \right)$, from (3.11), (3.20) and the fact that $\overline{T_N}$ is $\overline{G_N}$-invariant, we find that

$$\begin{aligned} &\overline{F}_n(x,y) - \overline{F}_n(x, \frac{p_d}{q_d}) = \overline{\lambda}\left(\left(\overline{T_N}\right)^{-n} (\Delta_{x,y}) \setminus \left(\overline{T_N}\right)^{-n} (\Delta_{x,p_d/q_d}) \right) \\ &\leq \frac{(N+1)^3}{N^2} \log\left(\frac{N+1}{N} \right) \overline{G_N}\left(\left(\overline{T_N}\right)^{-n} (\Delta_{x,y}) \setminus \left(\overline{T_N}\right)^{-n} (\Delta_{x,p_d/q_d}) \right) \\ &\leq \left(\frac{N+1}{N} \right)^3 \overline{\lambda}\left([0,x] \times \left[\frac{p_d}{q_d}, y \right] \right) \leq \left(\frac{N+1}{N} \right)^3 x \left| y - \frac{p_d}{q_d} \right| \\ &< \left(\frac{N+1}{N} \right)^3 \frac{(N)^{\frac{n}{N+1}+2} \cdot (N^2+4N) \cdot 2^{\frac{2n}{N+1}+2} \cdot x}{\left(N+\sqrt{N^2+4N}\right)^{\frac{2n}{N+1}+2}}. \end{aligned} \tag{3.21}$$

Since for every fixed $x \in [0,1]$, the function $y \mapsto \log\left(\frac{xy+N}{N}\right)$ is differentiable on $[0,1]$, by the *Mean Value Theorem* we have

$$\begin{aligned}\left|\log\left(\frac{xy+N}{N}\right)-\log\left(\frac{x\frac{p_d}{q_d}+N}{N}\right)\right| &= \left|y-\frac{p_d}{q_d}\right|\cdot\left|\frac{x}{x\xi+N}\right| \\ &\le x\cdot\left|y-\frac{p_d}{q_d}\right| < \frac{(N)^{\frac{n}{N+1}+2}\cdot(N^2+4N)\cdot 2^{\frac{2n}{N+1}+2}\cdot x}{\left(N+\sqrt{N^2+4N}\right)^{\frac{2n}{N+1}+2}}, \end{aligned} \tag{3.22}$$

where $p_d/q_d \le \xi \le y$.

By Lemma 3.1, (3.21) and (3.22), and the fact $\overline{F}_n\left(0,\frac{p_d}{q_d}\right)=0$, we have

$$\begin{aligned}&\left|\overline{F}_n(x,y)-\frac{1}{\log\left(\frac{N+1}{N}\right)}\log\left(\frac{xy+N}{N}\right)\right| \\ &\le \left|\overline{F}_n(x,y)-\overline{F}_n\left(x,\frac{p_d}{q_d}\right)\right|+\left|\overline{F}_n\left(x,\frac{p_d}{q_d}\right)-\overline{F}_n\left(0,\frac{p_d}{q_d}\right)-\frac{1}{\log\left(\frac{N+1}{N}\right)}\log\left(\frac{x\frac{p_d}{q_d}+N}{N}\right)\right| \\ &\quad+\frac{1}{\log\left(\frac{N+1}{N}\right)}\left|\log\left(\frac{xy+N}{N}\right)-\log\left(\frac{x\frac{p_d}{q_d}+N}{N}\right)\right| \\ &\le \left(\frac{N+1}{N}\right)^3\frac{(N)^{\frac{n}{N+1}+2}\cdot(N^2+4N)\cdot 2^{\frac{2n}{N+1}+2}\cdot x}{\left(N+\sqrt{N^2+4N}\right)^{\frac{2n}{N+1}+2}}+\frac{(N+1)^3}{N^2}\frac{C}{1-q}x\frac{p_d}{q_d}q^{n-d} \\ &\quad+\frac{1}{\log\left(\frac{N+1}{N}\right)}\frac{(N)^{\frac{n}{N+1}+2}\cdot(N^2+4N)\cdot 2^{\frac{2n}{N+1}+2}\cdot x}{\left(N+\sqrt{N^2+4N}\right)^{\frac{2n}{N+1}+2}} \\ &\le \left(\left(\frac{N+1}{N}\right)^3+\frac{1}{\log\left(\frac{N+1}{N}\right)}\right)\frac{(N)^{\frac{n}{N+1}+2}\cdot(N^2+4N)\cdot 2^{\frac{2n}{N+1}+2}}{\left(N+\sqrt{N^2+4N}\right)^{\frac{2n}{N+1}+2}}+\frac{(N+1)^3}{N^2}\frac{C}{1-q}q^{n-d}.\end{aligned}$$

Since $d \le n/(N+1)+2$, we have

$$\begin{aligned}&\left|\overline{F}_n(x,y)-\frac{\log\left(\frac{xy+N}{N}\right)}{\log\left(\frac{N+1}{N}\right)}\right| \le \left(\left(\frac{N+1}{N}\right)^3+\frac{1}{\log\left(\frac{N+1}{N}\right)}\right) \\ &\quad\times\frac{(N)^{\frac{n}{N+1}+2}\cdot(N^2+4N)\cdot 2^{\frac{2n}{N+1}+2}\cdot x}{\left(N+\sqrt{N^2+4N}\right)^{\frac{2n}{N+1}+2}}+\frac{(N+1)^3}{N^2}\frac{C}{(1-q)q^2}q^{n\frac{N}{N+1}} \le \mathcal{K}\rho^n,\end{aligned}$$

where

$$\mathcal{K}=\frac{4N^2(N^2+4N)}{(N+\sqrt{N^2+4N})^2}\left(\left(\frac{N+1}{N}\right)^3+\frac{1}{\log\left(\frac{N+1}{N}\right)}\right)+\frac{(N+1)^3}{N^2}\frac{C}{(1-q)q^2}$$

and

$$\rho = \max\left(q^{\frac{N}{N+1}}, \frac{(4N)^{\frac{1}{N+1}}}{(N+\sqrt{N^2+4N})^{\frac{2}{N+1}}} \right) < 1.$$

The proof is completed. □

4. Improving results

In this section, we shall estimate the error term

$$e_{n,t}(x,y) = G_{N,t}\left(T_N^n \in [0,x], s_{n,t} \in [0,y]\right) - \frac{1}{\log\left(\frac{N+1}{N}\right)} \log\left(\frac{xy+N}{N}\right)$$

for any $t, x, y \in I$ and $n \in \mathbb{N}$.

In the main result of this section, Theorem 4.4, we shall derive lower and upper bounds (not depending on $t \in I$) of the supremum

$$\sup_{x\in I, y\in I} |e_{n,t}(x,y)|, \quad t \in I,$$

which provide a more refined estimate of the convergence rate involved. First, we obtain a lower bound for the error, which suggests the exact convergence rate of $G_{N,t}\left(s_{n,t} \in [0,y]\right)$ to $G_N\left([0,y]\right)$ as $n \to \infty$ for all $t \in I$.

Theorem 4.1. *For any $t \in I$ and $n \in \mathbb{N}_+$, we have*

$$\frac{1}{2} v_{N(n)}(1) \le \sup_{y\in I} \left| G_{N,t}\left(s_{n,t} \in [0,y]\right) - G_N\left([0,y]\right) \right|$$

with $v_{N(n)}(t) = \sup_{s\in I} G_{N,t}\left(s_{n,t} = s\right)$, where we write $N(n)$ for $(i_1, \ldots, i_n)$ with $i_1 = \cdots = i_n = N$, $n \in \mathbb{N}_+$.

Proof. First, the continuity of the function $y \mapsto G_N\left([0,y]\right)$, $y \in I$, and the equations

$$\lim_{h\searrow 0} G_{N,t}\left(s_{n,t} \le y - h\right) = G_{N,t}\left(s_{n,t} < y\right)$$

and

$$\lim_{h\searrow 0} G_{N,t}\left(s_{n,t} < y + h\right) = G_{N,t}\left(s_{n,t} \le y\right)$$

imply that

$$\sup_{y\in I} \left| G_{N,t}\left(s_{n,t} \le y\right) - G_N\left([0,y]\right) \right| = \sup_{y\in I} \left| G_{N,t}\left(s_{n,t} < y\right) - G_N\left([0,y]\right) \right|$$

for all $t \in I$ and $n \in \mathbb{N}$. Second, whatever $s \in I$ we have,

$$\begin{aligned} G_{N,t}(s_{n,t}=s) &= G_{N,t}\left(s_{n,t} \le s\right) - G_N\left([0,s]\right) - \left(G_{N,t}\left(s_{n,t} < s\right) - G_N\left([0,s]\right)\right) \\ &\le \sup_{y\in I}|G_{N,t}(s_{n,t}\le y) - G_N([0,y])| + \sup_{y\in I}|G_{N,t}(s_{n,t}<y) - G_N([0,y])| \\ &= 2\sup_{y\in I}\left|G_{N,t}\left(s_{n,t}\le y\right) - G_N\left([0,y]\right)\right|. \end{aligned}$$

Hence,

$$\begin{aligned} &\sup_{y\in I}\left|G_{N,t}\left(s_{n,t}\in[0,y]\right) - G_N\left([0,y]\right)\right| \\ &\quad = \sup_{y\in I}\left|G_{N,t}\left(s_{n,t}\le y\right) - G_N\left([0,y]\right)\right| \ge \frac{1}{2}\sup_{s\in I} G_{N,t}\left(s_{n,t}=s\right), \end{aligned}$$

for all $t \in I$ and $n \in \mathbb{N}$.

By induction with respect to $n \in \mathbb{N}_+$, we get

$$U^n f(x) = \sum_{i_1,\dots,i_n\in\Lambda} v_{i_1\dots i_n}(x) f\left(u_{i_n\dots i_1}(x)\right), \tag{4.1}$$

where $u_{i_n\dots i_1} = u_{i_n}\circ\cdots\circ u_{i_1}$, $v_{i_1\dots i_n}(x) = v_{i_1}(x)v_{i_2}(u_{i_1}(x))\dots v_{i_n}(u_{i_{n-1}\dots i_1}(x))$, $n \ge 2$, and the functions u_i and v_i, $i \in \Lambda$, are defined in (2.20) and (2.21).

It follows from (2.22) that

$$U^n f(t) = E_t(f(s_{n,t})), \quad n \in \mathbb{N},\ f \in B(I),\ t \in I.$$

As $s_{n,t} = u_{a_n,\dots,a_1}(t)$, $t \in I$, $n \in \mathbb{N}_+$, we have

$$U^n f(t) = \sum_{i^{(n)}\in\Lambda^n} G_{N,t}\left((a_1,a_2,\dots,a_n) = i^{(n)}\right) f\left(u_{i_n\dots i_1}(t)\right) \tag{4.2}$$

for any $n \in \mathbb{N}_+$, $f \in B(I)$, $t \in I$ and $i^{(n)} = (i_1,\dots,i_n) \in \Lambda^n$.

Hence, by (2.7), (4.1) and (4.2), we get

$$v_{i_1\dots i_n}(t) = G_{N,t}\left(I_N\left(i^{(n)}\right)\right) = G_{N,t}\left(s_{n,t} = [i_n,\dots,i_2,i_1+t]_N\right), \quad n \ge 2,$$

$$v_{i_1}(t) = G_{N,t}\left(I_N\left(i_1\right)\right) = G_{N,t}\left(s_{1,t} = \frac{N}{i_1+t}\right),$$

for all $t \in I$ and $i_1,\dots,i_n \in \Lambda$.

Since, as easily seen,

$$\max_{i^{(n)}\in\Lambda^n} G_{N,t}\left(I_N\left(i^{(n)}\right)\right) = G_{N,t}\left(I_N\left(N(n)\right)\right),$$

where we write $N(n)$ for $i^{(n)} = (i_1, \dots, i_n)$ with $i_1 = \cdots = i_n = N$, $n \in \mathbb{N}_+$.

In [12], we showed that $I_N\left(i^{(n)}\right)$ is the set of irrationals in the interval with endpoints $\dfrac{p_n}{q_n}$ and $\dfrac{p_n + p_{n-1}}{q_n + q_{n-1}}$. Since

$$\frac{p_n}{q_n} = [i_1, \dots, i_n]_N = \begin{cases} \dfrac{N}{i_1} & \text{if } n = 1, \\ \dfrac{N}{i_1 + \dfrac{p_{n-1}(i_2, \dots, i_n)}{q_{n-1}(i_2, \dots, i_n)}} & \text{if } n \geq 2, \end{cases}$$

and

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}} = \begin{cases} \dfrac{N}{i_1 + 1} & \text{if } n = 1, \\ [i_1, \dots, i_{n-1}, i_n + 1]_N & \text{if } n \geq 2, \end{cases}$$

$$= \begin{cases} \dfrac{N}{i_1 + 1} & \text{if } n = 1, \\ \dfrac{N}{i_1 + \dfrac{p_n(i_2, \dots, i_n, N)}{q_n(i_2, \dots, i_n, N)}} & \text{if } n \geq 2, \end{cases}$$

we can write

$$\begin{aligned} v_{i_1 \dots i_n}(t) = {} & \frac{(t+N)N^{n-1}}{q_{n-1}(i_2, \dots, i_n)(t + i_1) + p_{n-1}(i_2, \dots, i_n)} \\ & \times \frac{1}{q_n(i_2, \dots, i_n, N)(t + i_1) + p_n(i_2, \dots, i_n, N)} \end{aligned} \tag{4.3}$$

for all $i_n \in \Lambda$, $n \geq 2$, and $t \in I$.

Also by (4.3), we have

$$\begin{aligned} v_{N(n)}(t) = {} & \frac{(t+N)N^{n-1}}{q_{n-1}(\underbrace{N, \dots, N}_{(n-1)\text{ times}})(t+N) + p_{n-1}(\underbrace{N, \dots, N}_{(n-1)\text{ times}})} \\ & \times \frac{1}{q_n(\underbrace{N, \dots, N, N}_{n\text{ times}})(t+N) + p_n(\underbrace{N, \dots, N, N}_{n\text{ times}})}. \end{aligned}$$

It is easy to see that $v_{N(n)}(\cdot)$ is a decreasing function. Therefore

$$\sup_{s\in I} G_{N,t}\left(s_{n,t}=s\right)=v_{N(n)}(t)\geq v_{N(n)}(1)$$

for all $t\in I$. □

Theorem 4.2 (the lower bound)**.** *For any $t\in I$, we have*

$$\frac{1}{2}v_{N(n)}(1)\leq \sup_{x\in I,y\in I}\left|G_{N,t}\left(T_N^n\in[0,x],s_{n,t}\in[0,y]\right)-\frac{\log\left(\frac{xy+N}{N}\right)}{\log\left(\frac{N+1}{N}\right)}\right|$$

for all $n\in\mathbb{N}_+$.

Proof. Whatever $t\in I$ and $n\in\mathbb{N}_+$, by Theorem 4.1 we have,

$$\begin{aligned}&\sup_{x\in I,y\in I}\left|G_{N,t}\left(T_N^n\in[0,x],s_{n,t}\in[0,y]\right)-\frac{1}{\log\left(\frac{N+1}{N}\right)}\log\left(\frac{xy+N}{N}\right)\right|\\&\geq\sup_{y\in I}\left|G_{N,t}\left(T_N^n\in I,s_{n,t}\in[0,y]\right)-\frac{1}{\log\left(\frac{N+1}{N}\right)}\log\left(\frac{y+N}{N}\right)\right|\\&=\sup_{y\in I}\left|G_{N,t}\left(s_{n,t}\in[0,y]\right)-G_N\left([0,y]\right)\right|\geq\frac{1}{2}v_{N(n)}(1).\end{aligned}$$

□

Remark 4.1. Note that

$$v_{N(n)}(1)=\frac{(1+N)N^{n+1}}{\overline{q}_{n+1}\overline{q}_{n+2}},\quad n\in\mathbb{N}_+,$$

where $\overline{q}_n$ is defined in (3.17). Using (3.19), it should be noted that Theorem 4.2 in connection with the limit

$$\lim_{n\to\infty}\left(\frac{1}{2}v_{N(n)}(1)\right)^{1/n}=\frac{2}{N+\sqrt{N^2+4N}+2}$$

leads to an estimate of the order of magnitude of the error $e_{n,t}(x,y)$.

It is known that for the RCF-expansion [7], the exact order of magnitude of the supremum $\sup_{x\in I,y\in I}\left|\mu_t\left(\tau^n\in[0,x],s_{n,t}\in[0,y]\right)-\frac{\log(xy+1)}{\log 2}\right|$, where τ denotes the Gauss map (1.2), $\tau^n=[a_{n+1},a_{n+2},\ldots]$ (= the regular continued fraction with incomplete quotients $a_{n+1},a_{n+2},\ldots$), $\mu_t([0,x])=\frac{(t+1)x}{tx+1}$, $x,t\in I$ and $s_{n+1,t}=1/(s_{n,t}+a_{n+1})$, $n\geq 0$, $s_{0,t}=t\in I$, is $\mathcal{O}(g^{2n})$ with $g=(\sqrt{5}-1)/2$, $g^2=(3-\sqrt{5})/2{=}0.38196\ldots$. Note that for $N{=}1$, we have $\lim_{n\to\infty}\left(\frac{1}{2}v_{N(n)}(1)\right)^{1/n}=g^2$.

In what follows, we study the transition operator associated to the RSCC underlying N-continued fraction on the Banach space of complex-valued functions of bounded variation. The characteristic properties of this operator are used to derive an explicit upper bound for $\sup\limits_{x\in I, y\in I} |e_{n,t}(x,y)|$, $t \in I$.

Let $BV(I)$ be the Banach space of complex-valued functions f of bounded variation on I under the norm

$$\|f\|_{\mathrm{v}} := \mathrm{var} f + |f|,$$

where $|f| := \sup_{x\in I} |f(x)|$. Remember that the variation $\mathrm{var}_A f$ over $A \subset I$ of $f \in B(I)$ is defined as

$$\sup \sum_{i=1}^{k} |f(\omega_i) - f(\omega_{i+1})|,$$

the supremum being taken over all $\omega_1 < \cdots < \omega_k \in A$, $k \geq 2$. We write simply $\mathrm{var} f$ for $\mathrm{var}_I f$, and if $\mathrm{var} f < \infty$, then f is called a function of bounded variation.

We start by proving the following elementary result.

Proposition 4.1. *For any $f \in BV(I)$, we have*

$$\mathrm{var}\, Uf \leq \frac{1}{N+1} \cdot \mathrm{var} f + K_N \cdot |f|,$$

where

$$K_N := \frac{2\sqrt{N^2+N}}{(N+\sqrt{N^2+N})(N+\sqrt{N^2+N}+1)}. \tag{4.4}$$

Proof. Recall that $v_i(x) = (x+N)\left(\dfrac{1}{x+i} - \dfrac{1}{x+i+1}\right)$, $i \in \Lambda$. We have

$$v_i'(x) = \frac{1}{(x+i)(x+i+1)}\left[1 - \frac{(x+N)(2x+2i+1)}{(x+i)(x+i+1)}\right] = \frac{L(N,x)}{(x+i)^2(x+i+1)^2}$$

with $L(N,x) = -x^2 - 2Nx + i(i+1) - N(2i+1)$, for every $i \in \Lambda$.

If $N \leq i \leq 2N-1$, then $L(N,x) < 0$ for all $x \in I$, i.e., $v_i'(x) < 0$, $x \in I$. Hence, the functions v_i are decreasing on I.

If $i = 2N$, then $L(N,x) > 0$ for all $x \in [0, -N+\sqrt{N^2+N}]$, i.e., $v_i'(x) > 0$, $x \in [0, -N+\sqrt{N^2+N}]$, and $L(N,x) < 0$ for all $x \in [-N+\sqrt{N^2+N}, 1]$, i.e., $v_i'(x) < 0$, $x \in [-N+\sqrt{N^2+N}, 1]$. Hence, v_{2N} is increasing on $[0, -N+\sqrt{N^2+N}]$ and decreasing on $[-N+\sqrt{N^2+N}, 1]$.

If $i \geq 2N+1$, then $L(N,x) > 0$ for all $x \in I$, i.e., $v_i'(x) > 0$, $x \in I$. Hence, the functions v_i are increasing on I.

Hence,

$$\operatorname{var} v_i = \begin{cases} v_i(0) - v_i(1), & N \le i \le 2N-1, \\ 2v_i(-N+\sqrt{N^2+N}) - v_i(0) - v_i(1), & i = 2N, \\ v_i(1) - v_i(0), & i \ge 2N+1, \end{cases}$$

and

$$|v_i| = \sup_{x\in I} v_i(x) = \begin{cases} v_i(0), & N \le i \le 2N-1, \\ v_i(-N+\sqrt{N^2+N}), & i = 2N, \\ v_i(1), & i \ge 2N+1. \end{cases}$$

Thus

$$\begin{aligned} \sup_{i\in\Lambda} |v_i| &= \max\left\{v_N(0), v_{2N}(-N+\sqrt{N^2+N}), v_{2N+1}(1)\right\} \\ &= \max\left\{\frac{1}{N+1}, \frac{\sqrt{N^2+N}}{(N+\sqrt{N^2+N})(N+\sqrt{N^2+N}+1)}, \frac{N+1}{(2N+2)(2N+3)}\right\} \\ &= \frac{1}{N+1}. \end{aligned}$$

Also,

$$\begin{aligned} \sum_{i\in\Lambda} \operatorname{var} v_i &= \sum_{N\le i\le 2N-1} (v_i(0) - v_i(1)) + \operatorname{var} v_{2N} + \sum_{i\ge 2N+1} (v_i(1) - v_i(0)) \\ &= \frac{1}{2(2N+1)} + \frac{2\sqrt{N^2+N}}{(N+\sqrt{N^2+N})(N+\sqrt{N^2+N}+1)} - \frac{1}{2N+1} + \frac{1}{2(2N+1)} \\ &= \frac{2\sqrt{N^2+N}}{(N+\sqrt{N^2+N})(N+\sqrt{N^2+N}+1)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \operatorname{var} Uf &= \operatorname{var} \sum_{i\in\Lambda} v_i \cdot (f\circ u_i) \le \sum_{i\in\Lambda} \operatorname{var}(v_i \cdot (f\circ u_i)) \\ &\le \sum_{i\in\Lambda} |v_i| \operatorname{var}(f\circ u_i) + \sum_{i\in\Lambda} |f\circ u_i| \operatorname{var} v_i \\ &\le \left(\sup_{i\in\Lambda} |v_i|\right) \sum_{i\in\Lambda} \operatorname{var}(f\circ u_i) + |f| \sum_{i\in\Lambda} \operatorname{var} v_i \le \frac{1}{N+1} \cdot \operatorname{var} f + K_N \cdot |f|, \end{aligned}$$

where the constant K_N is as in (4.4), and because we took into account that

$$\sum_{i\in\Lambda} \operatorname{var}(f\circ u_i) = \sum_{i\in\Lambda} \operatorname{var}_{\left[\frac{N}{i+1}, \frac{N}{i}\right]} f = \operatorname{var} f. \qquad \square$$

Corollary 4.1. *For any $f \in BV(I)$ and for all $n \in \mathbb{N}$, we have*

$$\operatorname{var} U^n f \le \left(\frac{1}{N+1} + K_N\right)^n \cdot \operatorname{var} f, \tag{4.5}$$

$$|U^n f - U^\infty f| \le \left(\frac{1}{N+1} + K_N\right)^n \cdot \operatorname{var} f. \tag{4.6}$$

Proof. Note that for any $f \in BV(I)$ and $u \in I$, since $\int_I \mathrm{d}G_N(x) = 1$, we have

$$\begin{aligned} |f(u)| - \left|\int_I f(x)\mathrm{d}G_N(x)\right| &\le \left|f(u) - \int_I f(x)\mathrm{d}G_N(x)\right| \\ &= \left|\int_I (f(u) - f(x))\mathrm{d}G_N(x)\right| \le \operatorname{var} f, \end{aligned}$$

whence

$$|f| = \sup_{u\in I} |f(u)| \le \left|\int_I f(x)\mathrm{d}G_N(x)\right| + \operatorname{var} f, \quad f \in BV(I). \tag{4.7}$$

Finally, (2.23), (2.24) and (4.7) imply that

$$\begin{aligned} |U^n f - U^\infty f| &\le \left|\int_I \left(U^n f - U^\infty f\right)(x)\mathrm{d}G_N(x)\right| + \operatorname{var}\left(U^n f - U^\infty f\right) \\ &\le |U^\infty U^n f - U^\infty f| + \operatorname{var} U^n f = \operatorname{var} U^n f, \end{aligned} \tag{4.8}$$

for all $n \in \mathbb{N}$ and $f \in BV(I)$. Remind that $U^\infty f$ is a constant.

It follows from Proposition 4.1 that for all $f \in BV(I)$, we have

$$\operatorname{var}\left(Uf - U^\infty f\right) \le \frac{1}{N+1} \cdot \operatorname{var}\left(f - U^\infty f\right) + K_N \cdot \left|f - U^\infty f\right|.$$

But,

$$\operatorname{var}\left(Uf - U^\infty f\right) = \operatorname{var} Uf, \qquad \operatorname{var}\left(f - U^\infty f\right) = \operatorname{var} f,$$

and $|f - U^\infty f| \le \operatorname{var} f$, which is (4.8) with $n = 0$. Thus,

$$\operatorname{var} Uf \le \frac{1}{N+1} \cdot \operatorname{var} f + K_N \cdot \operatorname{var} f = \left(\frac{1}{N+1} + K_N\right) \cdot \operatorname{var} f,$$

which leads to (4.5). Next, (4.6) follows from (4.8) and (4.5). □

Theorem 4.3 (the upper bound)**.** *For any $t \in I$, we have*

$$\sup_{x\in I, y\in I} \left| G_{N,t}\left(T_N^n \in [0,x], s_{n,t} \in [0,y]\right) - \frac{\log\left(\frac{xy+N}{N}\right)}{\log\left(\frac{N+1}{N}\right)} \right| \le \left(\frac{1}{N+1} + K_N\right)^n,$$

for all $n \in \mathbb{N}$.

PROOF. Let $F_{n,t}(y) = G_{N,t}(s_{n,t} \leq y)$ and $H_{n,t}(y) = F_{n,t}(y) - G_N([0,y])$, $t, y \in I$, $n \in \mathbb{N}$. Note that $H_{n,t}(0) = 0$. As we have noted, U is the transition operator of the Markov chain $(s_{n,t})_{n\in\mathbb{N}}$. For any $y \in I$, consider the function f_y defined on I as

$$f_y(t) := \begin{cases} 1 & \text{if } 0 \leq t \leq y, \\ 0 & \text{if } y < t \leq 1. \end{cases}$$

Hence

$$U^n f_y(t) = E_t\,(f_y(s_{n,t})|\, s_{0,t} = t) = G_{N,t}(s_{n,t} \leq y)$$

for all $t, y \in I$, $n \in \mathbb{N}$. As

$$U^\infty f_y = \int_I f_y(t)\mathrm{d}G_N(t) = G_N([0,y]), \quad y \in I.$$

It follows from Corollary 4.1 that

$$\begin{aligned} |H_{n,t}(y)| &= |G_{N,t}(s_{n,t} \leq y) - G_N([0,y])| \\ &= |U^n f_y(t) - U^\infty f_y| \leq \left(\frac{1}{N+1} + K_N\right)^n \operatorname{var} f_y = \left(\frac{1}{N+1} + K_N\right)^n \end{aligned} \tag{4.9}$$

for all $t, y \in I$, $n \in \mathbb{N}$. By the very definition of the conditional probability and (2.25), for all $t \in I$, $x, y \in I$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} &G_{N,t}\,(T_N^n \in [0,x], s_{n,t} \in [0,y]) \\ &\quad = G_{N,t}\,(T_N^n \in [0,x] \mid\, s_{n,t} \in [0,y]) \cdot G_{N,t}(s_{N,t} \in [0,y]) \\ &\quad = G_{N,t}\,(T_N^n \in [0,x] \mid\, s_{n,t} \in [0,y]) \cdot F_{n,t}(y) \\ &\quad = \int_0^y G_{N,t}\,(T_N^n \in [0,x] \mid\, s_{n,t} = z)\,\mathrm{d}F_{n,t}(z) \quad = \int_0^y \frac{(z+N)x}{zx+N}\mathrm{d}F_{n,t}(z) \\ &\quad = \int_0^y \frac{(z+N)x}{zx+N}\mathrm{d}G_N(z) + \int_0^y \frac{(z+N)x}{zx+N}\mathrm{d}H_{n,t}(z) \\ &\quad = \frac{1}{\log\left(\frac{N+1}{N}\right)} \log\left(\frac{xy+N}{N}\right) + \frac{(z+N)x}{zx+N} H_{n,t}(z)|_0^y - \int_0^y \frac{Nx(1-x)}{(zx+N)^2} H_{n,t}(z)\mathrm{dz}. \end{aligned}$$

Hence, by (4.9)

$$\begin{aligned} &\left| G_{N,t}\,(T_N^n \in [0,x], s_{n,t} \in [0,y]) - \frac{1}{\log\left(\frac{N+1}{N}\right)} \log\left(\frac{xy+N}{N}\right) \right| \\ &\qquad \leq \left(\frac{1}{N+1} + K_N\right)^n \left(\frac{(y+N)x}{xy+N} + \frac{N(1-x)}{zx+N}\Big|_{z=0}^{z=y} \right) \end{aligned}$$

$$= \left(\frac{1}{N+1} + K_N\right)^n \left(\frac{(y+N)x}{xy+N} - \frac{(1-x)xy}{xy+N}\right)$$
$$= \left(\frac{1}{N+1} + K_N\right)^n \cdot x \le \left(\frac{1}{N+1} + K_N\right)^n,$$

where K_N is as in (4.4), $t, x, y \in I$, $n \in \mathbb{N}$. □

Combining Theorems 4.2 and 4.3, we obtain Theorem 4.4.

Theorem 4.4. *Whatever $t \in I$ we have,*

$$\frac{1}{2} v_{N(n)}(1) \le \sup_{x\in I, y\in I} \left| G_{N,t}\left(T_N^n \in [0,x], s_{n,t} \in [0,y]\right) - \frac{\log\left(\frac{xy+N}{N}\right)}{\log\left(\frac{N+1}{N}\right)} \right| \le \left(\frac{1}{N+1} + K_N\right)^n$$

for all $n \in \mathbb{N}_+$.

Remark 4.2. Theorem 4.4 implies that the convergence rate is $\mathcal{O}(\alpha^n)$, with

$$\frac{2}{N + \sqrt{N^2+4N} + 2} \le \alpha \le \frac{1}{N+1} + K_N.$$

For example, we have

$N = 1$	$g^2 = 0.381966 \le \alpha \le 0.843146$
$N = 2$	$0.267949192 \le \alpha \le 0.535374333\ldots$
$N = 5$	$0.145898033 \le \alpha \le 0.257764367\ldots$
$N = 10$	$0.083920216 \le \alpha \le 0.138555191\ldots$
$N = 100$	$0.009804864 \le \alpha \le 0.01487615$
$N = 1000$	$0.000998004 \le \alpha \le 0.001498751$

Table 1. Lower and upper bounds for some $N \ge 1$.

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(Received December 17, 2018; revised November 8, 2019)

Periodica Mathematica Hungarica
https://doi.org/10.1007/s10998-020-00325-2

Convergence rate for Rényi-type continued fraction expansions

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Abstract
This paper continues our investigation of Rényi-type continued fractions studied in Lascu and Sebe (A dependence with complete connections approach to generalized Rényi continued fractions, Acta Math. Hungar. 160, 292–313, 2020). A Wirsing-type approach to the Perron–Frobenius operator of the Rényi-type continued fraction transformation under its invariant measure allows us to study the optimality of the convergence rate. Actually, we obtain upper and lower bounds of the convergence rate which provide a near-optimal solution to the Gauss–Kuzmin–Lévy problem.

Keywords Rényi continued fractions · Perron–Frobenius operator · Invariant measure · Gauss–Kuzmin–Lévy problem

Mathematics Subject Classification Primary 11J70 · 11K50; Secondary 47B38 · 60F05

1 Introduction

In [6] we showed that every $x \in [0, 1]$ can be written as

$$x = 1 - \cfrac{N}{1 + a_1 - \cfrac{N}{1 + a_2 - \cfrac{N}{1 + a_3 - \ddots}}} := [a_1, a_2, a_3, \ldots]_R, \tag{1.1}$$

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Published online: 16 March 2020

where N is a fixed integer greater than or equal to 2, and a_n's are positive integers greater than or equal to N.

For a fixed integer $N \geq 2$, we define the shift transformation $R_N : [0, 1] \to [0, 1]$, by $R_N(1) = 0$ and $R_N(x) = R_N([a_1, a_2, a_3, \ldots]_R) := [a_2, a_3, a_4, \ldots]_R$ for $x \neq 1$. There is another way to define R_N for $x \in [0, 1]$. Denoting the floor function by $\lfloor \cdot \rfloor$ we have

$$R_N(x) := \begin{cases} \dfrac{N}{1-x} - \left\lfloor \dfrac{N}{1-x} \right\rfloor, & x \in [0, 1) \\ 0, & x = 1. \end{cases} \tag{1.2}$$

Since the case $N = 1$ refers to the Rényi interval map, we call the transformation in (1.2) *Rényi-type continued fraction transformation*.

The digits or incomplete quotients of x with respect to the Rényi-type continued fraction expansion are defined by

$$a_n := a_n(x) = a_1\left(R_N^{n-1}(x)\right), \quad n \geq 2, \tag{1.3}$$

with $R_N^0(x) = x$ and

$$a_1 := a_1(x) = \begin{cases} \left\lfloor \frac{N}{1-x} \right\rfloor & \text{if } x \neq 1, \\ \infty & \text{if } x = 1. \end{cases} \tag{1.4}$$

These transformations belong to a wider one parameter family of interval maps of the form $T_u(x) := \frac{1}{u(1-x)} - \lfloor \frac{1}{u(1-x)} \rfloor$, where $u > 0, x \in [0, 1)$. As the parameter varies in $(0, 4)$ there is a viable theory of a one parameter family of continued fractions, which fails when $u \geq 4$. Named *u-backward continued fractions*, they possess some attractive properties which are not shared by the regular continued fractions. A natural question was whether the dynamical system given by the maps T_u admits an absolutely continuous invariant measure. Grochenig and Haas showed in [3] that the invariant measure for T_u is finite if and only if $0 < u < 4$ and $u = u_q \neq 4\cos^2\frac{\pi}{q}$, $q = 3, 4, \ldots$. They have identified that for certain values of u (for example, if $u = 1/N$ for positive integers $N \geq 2$) $R_N := T_{1/N}$ has a unique absolutely continuous invariant measure

$$\rho_N(A) := \frac{1}{\log\left(\frac{N}{N-1}\right)} \int_A \frac{\mathrm{d}x}{x + N - 1}, \quad A \in \mathcal{B}_{[0,1]}, \tag{1.5}$$

where $\mathcal{B}_{[0,1]}$ denotes the σ-algebra of all Borel subsets of $[0, 1]$.

It was proved in [3] that the dynamical system $([0, 1], R_N, \rho_N)$ is ergodic. Using the ergodicity of R_N and Birkhoff's ergodic theorem [2] a number of results were obtained in [3,4]. It should be stressed that the ergodic theorem does not yield any information on the convergence rate in the Gauss–Kuzmin problem that amounts to the asymptotic behaviour of $\mu\left(R_N^{-n}\right)$ as $n \to \infty$, where μ is an arbitrary probability measure on $\mathcal{B}_{[0,1]}$. So that a Gauss–Kuzmin theorem is needed.

Recently, in [6] we proved a version of a Gauss–Kuzmin theorem. Using the natural extension for Rényi-type continued fraction expansions, we obtained an infinite-order-chain representation of the sequence of the incomplete quotients of these expansions. Together with the ergodic behaviour of a certain homogeneous random system with complete connections this allowed us to solve a variant of the Gauss–Kuzmin problem. Following the treatment in the case of the regular continued fractions [5], the Gauss–Kuzmin–Lévy problem for the transformation R_N can be approached in terms of the associated Perron–Frobenius operator. In Sect. 2 we focus our study on the Perron–Frobenius operator under the invariant measure

induced by the limit distribution function. In Sect. 3 we use a Wirsing-type approach [7] to get close to the optimal convergence rate. By restricting the domain of the Perron–Frobenius operator of R_N under its invariant measure ρ_N to the Banach space of functions which have a continuous derivative on $[0, 1]$, we obtain upper and lower bounds of the error which provide a refined estimate of the convergence rate. The last section gives some interesting numerical calculations.

2 Operator-theoretical treatment

Let $\left([0, 1], \mathcal{B}_{[0,1]}, R_N, \rho_N\right)$ be as in Sect. 1. In the sequel, we derive its Perron–Frobenius operator. Let μ be a probability measure on $\left([0, 1], \mathcal{B}_{[0,1]}\right)$ such that $\mu\left(R_N^{-1}(A)\right) = 0$ whenever $\mu(A) = 0$ for $A \in \mathcal{B}_{[0,1]}$, i.e., the transformation R_N is μ-*non-singular*. For example, this condition is satisfied if R_N is μ-preserving, that is, $\mu R_N^{-1} = \mu$.

The *Perron–Frobenius operator* U associated with R_N is defined as the bounded linear and positive operator on the Banach space $L^1([0, 1], \mu) := \{f : [0, 1] \to \mathbb{C} : \int_0^1 |f| \, d\mu < \infty\}$ which satisfies

$$\int_A Uf \, d\mu = \int_{R_N^{-1}(A)} f \, d\mu \quad \text{for all } f \in L^1([0, 1], \mu),\ A \in \mathcal{B}_{[0,1]}, \tag{2.1}$$

or, equivalently

$$\int_0^1 gUf \, d\mu = \int_0^1 (g \circ R_N) f \, d\mu \quad \text{for all } f \in L^1([0, 1], \mu) \text{ and } g \in L^\infty([0, 1]). \tag{2.2}$$

Here $L^\infty([0, 1])$ denotes the Banach space of λ-essentially bounded functions defined on $[0, 1]$, where λ is the Lebesgue measure. The existence and uniqueness of Uf follows from the Radon-Nikodym theorem. This implies also the existence and uniqueness of U.

In particular, the Perron–Frobenius operator U of R_N under the Lebesgue measure is given as follows ([1], p. 86):

$$Uf(x) = \frac{d}{dx} \int_{R_N^{-1}([0,x])} f(t) \, dt = \sum_{t \in R_N^{-1}(x)} \frac{f(t)}{|(R_N(t))'|} \quad \text{a.e. in } [0, 1]. \tag{2.3}$$

The following property is useful to prove the next proposition.

There exists $f \in L^1([0, 1], \mu)$ such that $f \geq 0$ and $Uf = f$ a.e. if and only if R_N preserves the measure ν which is defined as $\nu(A) := \int_A f d\mu$ for $A \in \mathcal{B}_{[0,1]}$. In particular, $U1 = 1$ if and only if R_N is μ-preserving ([1], p. 80).

Proposition 2.1 *The measure ρ_N in (1.5) is invariant under R_N in (1.2).*

Proof Let ν_N the density measure of ρ_N, i.e, $\nu_N(x) := \frac{1}{\log\left(\frac{N}{N-1}\right)} \frac{1}{x+N-1}$. From above, it is sufficient to show that the function ν_N is an eigenfunction of the Perron–Frobenius operator of R_N with the eigenvalue 1:

$$U\nu_N(x) = \sum_{t \in R_N^{-1}(x)} \frac{\nu_N(t)}{|(R_N(t))'|}. \tag{2.4}$$

First, we note that $R_N^{-1}(x) = \left\{1 - \frac{N}{x+i} : i \geq N, x \in [0,1]\right\}$. Thus,

$$U v_N(x) = \sum_{i \geq N} \frac{N}{(x+i)^2} v_N\left(1 - \frac{N}{x+i}\right) = \frac{1}{\log\left(\frac{N}{N-1}\right)} \sum_{i \geq N} \frac{1}{(x+i)(x+i-1)}$$
$$= \frac{1}{\log\left(\frac{N}{N-1}\right)} \frac{1}{x+N-1} = v_N(x).$$

□

Proposition 2.2 *Let $\left([0,1], \mathcal{B}_{[0,1]}, R_N, \rho_N\right)$ be as in Sect. 2, and let U denote its Perron–Frobenius operator. Then the following holds:*

(i) *The following equation holds:*

$$Uf(x) = \sum_{i \geq N} P_{N,i}(x)\, f\left(u_{N,i}(x)\right), \quad f \in L^1([0,1], \rho_N), \tag{2.5}$$

where $P_{N,i}$ and $u_{N,i}$ are functions defined on $[0,1]$ by

$$P_{N,i}(x) := \frac{x+N-1}{(x+i)(x+i-1)} \tag{2.6}$$

and

$$u_{N,i}(x) := 1 - \frac{N}{x+i}. \tag{2.7}$$

(ii) *Let μ be a probability measure on $\left([0,1], \mathcal{B}_{[0,1]}\right)$ such that $\mu \ll \lambda$, i.e., μ is absolutely continuous with respect to the Lebesgue measure λ and let $h := \mathrm{d}\mu/\mathrm{d}\lambda$ a.e. in $[0,1]$. Then for any $n \in \mathbb{N}_+ := \{1,2,3,\ldots\}$ and $A \in \mathcal{B}_{[0,1]}$, we have*

$$\mu\left(R_N^{-n}(A)\right) = \int_A U^n f(x) \mathrm{d}\rho_N(x) \tag{2.8}$$

where $f(x) := \left(\log\left(\frac{N}{N-1}\right)\right)(x+N-1)h(x)$ for $x \in [0,1]$.

Proof (i) Let $R_{N,i}$ denote the restriction of R_N to the subinterval $I(i) := \left(1 - \frac{N}{i}, 1 - \frac{N}{i+1}\right]$, $i \geq N$, that is,

$$R_{N,i}(x) = \frac{N}{1-x} - i, \quad x \in I(i). \tag{2.9}$$

Let $C(A) := (R_N)^{-1}(A)$ and $C_i(A) := \left(R_{N,i}\right)^{-1}(A)$ for $A \in \mathcal{B}_{[0,1]}$. Since $C(A) = \bigcup_i C_i(A)$ and $C_i \cap C_j$ is a null set when $i \neq j$, we have

$$\int_{C(A)} f \,\mathrm{d}\rho_N = \sum_{i \geq N} \int_{C_i(A)} f \,\mathrm{d}\rho_N, \quad f \in L^1([0,1], \rho_N),\ A \in \mathcal{B}_{[0,1]}. \tag{2.10}$$

For any $i \geq N$, by the change of variable $x = \left(R_{N,i}\right)^{-1}(y) = u_{N,i}(y)$, we successively obtain

$$\begin{aligned}\int_{C_i(A)} f(x)\,\rho_N(\mathrm{d}x) &= \left(\log\left(\frac{N}{N-1}\right)\right)^{-1} \int_{C_i(A)} \frac{f(x)}{x+N-1}\,\mathrm{d}x \\ &= \left(\log\left(\frac{N}{N-1}\right)\right)^{-1} \int_A \frac{1}{(y+i)(y+i-1)} f\left(u_{N,i}(y)\right)\mathrm{d}y \\ &= \int_A P_{N,i}(y)\, f\left(u_{N,i}(y)\right)\,\rho_N(\mathrm{d}y). \end{aligned} \tag{2.11}$$

Now, (2.5) follows from (2.10) and (2.11).
(ii) We will use mathematical induction. For $n = 0$, the Eq. (2.8) holds by definitions of f and h. Assume that (2.8) holds for some $n \in \mathbb{N}$. Then

$$\mu\left(R_N^{-(n+1)}(A)\right) = \mu\left(R_N^{-n}\left(R_N^{-1}(A)\right)\right) = \int_{C(A)} U^n f(x)\,\rho_N(\mathrm{d}x), \tag{2.12}$$

and by definition, we have

$$\int_{C(A)} U^n f(x)\,\rho_N(\mathrm{d}x) = \int_A U^{n+1} f(x)\,\rho_N(\mathrm{d}x). \tag{2.13}$$

Therefore,

$$\mu\left(R_N^{-(n+1)}(A)\right) = \int_A U^{n+1} f(x)\rho_N(\mathrm{d}x) \tag{2.14}$$

which completes the proof. □

Remark 2.3 Under the hypotheses of Proposition 2.2(ii) it follows that

$$\mu(R_N^{-n}(A)) - \rho_N(A) = \int_A (U^n f(x) - 1)\mathrm{d}\rho_N(x), \tag{2.15}$$

for any $n \in \mathbb{N}$ and $A \in \mathcal{B}_{[0,1]}$, where $f(x) := \left(\log\left(\frac{N}{N-1}\right)\right)(x+N-1)h(x)$, $x \in [0,1]$. The last equation shows that the asymptotic behavior of $\mu(R_N^{-n}(A)) - \rho_N(A)$ as $n \to \infty$ is given by the asymptotic behavior of the n-th power of the Perron–Frobenius operator U on $L^1([0,1], \rho_N)$.

3 Near-optimal solution to the Gauss–Kuzmin–Lévy problem

In this section we develop a Wirsing-type approach [7] to obtain a solution to the Gauss–Kuzmin–Lévy problem in Theorem 3.4.

Let μ be a probability measure on $\mathcal{B}_{[0,1]}$ such that $\mu \ll \lambda$. For any $n \in \mathbb{N}$ put $F_N^n(x) = \mu\left(R_N^n < x\right)$, $x \in [0,1]$, where R_N^0 is the identity map. As $\left(R_N^n < x\right) = R_N^{-n}((0,x))$, by Proposition 2.2 (ii) we have

$$F_N^n(x) = \int_0^x \frac{U^n f_N^0(u)}{u+N-1}\mathrm{d}u, \quad n \in \mathbb{N}, \tag{3.1}$$

where $f_N^0(x) := (x+N-1)\left(F_N^0\right)'(x)$, $x \in [0,1]$, where $\left(F_N^0\right)' = \mathrm{d}\mu/\mathrm{d}\lambda$.

We will assume that $\left(F_N^0\right)' \in C^1([0,1])$, the collection of all functions $f : [0,1] \to \mathbb{C}$ which have a continuous derivative. So, we study the behavior of U^n as $n \to \infty$, assuming that the domain of U is $C^1([0,1])$.

Let $f \in C^1([0,1])$. Then the series (2.5) can be differentiated term-by-term, since the series of derivatives is uniformly convergent. Next, since

$$P_{N,i}(x) = \frac{i+1-N}{x+i} - \frac{i-N}{x+i-1},$$

we get

$$\begin{aligned}(Uf)'(x) &= \sum_{i\geq N}\left\{(P_{N,i})'(x) f\left(u_{N,i}(x)\right) + P_{N,i}(x) f'\left(u_{N,i}(x)\right)\left(u_{N,i}\right)'(x)\right\} \\ &= \sum_{i\geq N}\left\{\left(\frac{i-N}{(x+i-1)^2} - \frac{i+1-N}{(x+i)^2}\right) f\left(u_{N,i}(x)\right) + P_{N,i}(x) f'\left(u_{N,i}(x)\right)\frac{N}{(x+i)^2}\right\} \\ &= \sum_{i\geq N}\left\{\frac{i+1-N}{(x+i)^2}\left[f\left(u_{N,i+1}(x)\right) - f\left(u_{N,i}(x)\right)\right] + P_{N,i}(x) f'\left(u_{N,i}(x)\right)\frac{N}{(x+i)^2}\right\},\end{aligned} \tag{3.2}$$

for any $x \in [0,1]$. Thus, we can write

$$(Uf)' = -Vf', \quad f \in C^1([0,1]), \tag{3.3}$$

where $V : C([0,1]) \to C([0,1])$ is defined by

$$Vg(x) = -\sum_{i\geq N}\left\{\frac{i+1-N}{(x+i)^2}\int_{u_{N,i}(x)}^{u_{N,i+1}(x)} g(u)\mathrm{d}u + \frac{N(x+N-1)}{(x+i-1)(x+i)^3} g\left(u_{N,i}(x)\right)\right\} \tag{3.4}$$

with $g \in C([0,1])$ and $x \in [0,1]$. Clearly, $(U^n f)' = (-1)^n V^n f'$, $n \in \mathbb{N}_+$, $f \in C^1([0,1])$. We are going to show that V^n takes certain functions into functions with very small values when $n \in \mathbb{N}_+$ is large.

Proposition 3.1 *For a fixed integer $N \geq 2$ there are positive constants $v_N < w_N < 1$ and a real-valued non-positive function $\varphi_N \in C([0,1])$ such that*

$$v_N(-\varphi_N) \leq V\varphi_N \leq w_N(-\varphi_N). \tag{3.5}$$

Proof For $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$, let $h_N : \mathbb{R}_+ \to \mathbb{R}$ be a continuous bounded function such that $\lim_{x\to\infty} h_N(x) < \infty$. We look for a function $g_N : (0,1] \to \mathbb{R}$ such that $Ug_N = h_N$, assuming that the equation

$$Ug_N(x) = \sum_{i\geq N} P_{N,i}(x) g_N\left(u_{N,i}(x)\right) = h_N(x) \tag{3.6}$$

holds for $x \in \mathbb{R}_+$. Reducing the terms of the series involved in (3.6) yields

$$\frac{h_N(x)}{x+N-1} - \frac{h_N(x+1)}{x+N} = \frac{1}{(x+N-1)(x+N)} g_N\left(\frac{x}{x+N}\right), \quad x \in \mathbb{R}_+. \tag{3.7}$$

Hence

$$g_N(u) = \frac{N}{1-u} h_N\left(\frac{Nu}{1-u}\right) - \left(\frac{N}{1-u} - 1\right) h_N\left(\frac{Nu}{1-u} + 1\right), \quad u \in (0,1], \tag{3.8}$$

and we indeed have $Ug_N = h_N$ since

$$\begin{aligned} Ug_N(x) &= \sum_{i\geq N} \frac{x+N-1}{(x+i-1)(x+i)} g_N\left(1-\frac{N}{x+i}\right) \\ &= (x+N-1)\sum_{i\geq N}\left(\frac{1}{x+i-1}-\frac{1}{x+i}\right) g_N\left(1-\frac{N}{x+i}\right) \\ &= (x+N-1)\sum_{i\geq N}\left(\frac{h_N(x+i-N)}{x+i-1}-\frac{h_N(x+i+1-N)}{x+i}\right) \\ &= (x+N-1)\left(\frac{h_N(x)}{x+N-1}-\lim_{i\to\infty}\frac{h_N(x+i+1-N)}{x+i}\right) \\ &= h_N(x), \quad x\in\mathbb{R}_+. \end{aligned} \tag{3.9}$$

In particular, for any fixed $t_N \in [0,1]$ we consider the function $h_{N,t_N} : \mathbb{R}_+ \to \mathbb{R}$ defined as

$$h_{N,t_N}(x) = \frac{1}{e_N x + t_N + 1}, \quad x \in \mathbb{R}_+ \tag{3.10}$$

where the coefficient e_N will be specified later. By the above, the function $g_{N,t_N} : (0,1] \to \mathbb{R}$ defined by

$$\begin{aligned} g_{N,t_N}(x) &= \frac{N}{1-x} h_{N,t_N}\left(\frac{Nx}{1-x}\right) - \left(\frac{N}{1-x}-1\right) h_{N,t_N}\left(\frac{Nx}{1-x}+1\right) \\ &= \frac{N}{e_N Nx + (t_N+1)(1-x)} - \frac{N-1+x}{e_N(Nx+1-x)+(t_N+1)(1-x)} \end{aligned} \tag{3.11}$$

satisfies

$$Ug_{N,t_N}(x) = h_{N,t_N}(x), \quad x \in [0,1]. \tag{3.12}$$

Setting

$$\begin{aligned} &\varphi_{N,t_N}(x) := (g_{N,t_N})'(x) \\ &= \frac{-N(e_N N - t_N - 1)}{\{e_N Nx + (t_N+1)(1-x)\}^2} - \frac{N\{2e_N - (e_N N - t_N - 1)\}}{\{e_N(Nx+1-x)+(t_N+1)(1-x)\}^2}, \end{aligned} \tag{3.13}$$

we have

$$V\varphi_{N,t_N}(x) = -(Ug_{N,t_N})'(x) = -(h_{N,t_N})'(x) = \frac{e_N}{(e_N x + t_N + 1)^2}, \quad x \in [0,1]. \tag{3.14}$$

Since g_{N,t_N} is a decreasing function it follows that $\varphi_{N,t_N}(x) < 0, x \in [0,1]$. Also, V is a linear operator that takes non-positive functions into positive functions. Therefore, $V\varphi_{N,t_N}(x) > 0$, $x \in [0,1]$.

We choose t_N by asking that $(\varphi_{N,t_N}/V\varphi_{N,t_N})(0) = (\varphi_{N,t_N}/V\varphi_{N,t_N})(1)$. Since

$$(\varphi_{N,t_N}/V\varphi_{N,t_N})(0) = \frac{(t_N+1)^2}{e_N}\left\{\frac{-N(e_N N - t_N - 1)}{(t_N+1)^2} - \frac{N\{2e_N - (e_N N - t_N - 1)\}}{(e_N + t_N + 1)^2}\right\} \tag{3.15}$$

and

$$(\varphi_{N,t_N}/V\varphi_{N,t_N})(1) = \frac{-2(e_N + t_N + 1)^2}{e_N^2 N} \tag{3.16}$$

this amounts to the equation

$$H_N(t_N) = 2(t_N + e_N + 1)^4 + e_N^3 N^2(1 - 2N)(t_N + 1) - e_N^4 N^3 = 0. \tag{3.17}$$

We choose the coefficient e_N such that the equation $H_N(x) = 0$, $x \in [0, 1]$, yields a unique solution $t_N \in [0, 1]$. Asking that

$$H_N(0) < 0, \quad H_N(1) > 0, \quad \text{and} \quad \frac{\mathrm{d}H_N}{\mathrm{d}t_N} > 0, \tag{3.18}$$

we may determine e_N (see "Appendix"). For this unique acceptable solution $t_N \in [0, 1]$ the function $\varphi_{N,t_N}/V\varphi_{N,t_N}$ attains its minimum equal to $-2(e_N + t_N + 1)^2/\left(e_N^2 N\right)$ at $x = 0$ and $x = 1$, and has a maximum $m(t_N) = \varphi_{N,t_N}/V\varphi_{N,t_N}(x_{max}) < 0$. It follows that

$$\frac{-2(e_N + t_N + 1)^2}{e_N^2 N} \le \frac{\varphi_{N,t_N}}{V\varphi_{N,t_N}} \le m(t_N).$$

Since $\dfrac{\varphi_{N,t_N}}{V\varphi_{N,t_N}} < 0$ we get

$$-m(t_N) \le \frac{-\varphi_{N,t_N}}{V\varphi_{N,t_N}} \le \frac{2(e_N + t_N + 1)^2}{e_N^2 N}.$$

Therefore,

$$\frac{e_N^2 N}{2(e_N + t_N + 1)^2}(-\varphi_{N,t_N}) \le V\varphi_{N,t_N} \le -\frac{1}{m(t_N)}(-\varphi_{N,t_N}).$$

It follows that for $\varphi_N = \varphi_{N,t_N}$ we have

$$v_N(-\varphi_N) \le V\varphi_N \le w_N(-\varphi_N),$$

where

$$v_N = \frac{e_N^2 N}{2(e_N + t_N + 1)^2} \quad \text{and} \quad w_N = -\frac{1}{m(t_N)}.$$

□

Remark 3.2 By (3.5) we successively get

$$\begin{aligned}
&v_N^2(-\varphi_N) \le -V^2\varphi_N \le w_N^2(-\varphi_N)\\
&v_N^3(-\varphi_N) \le V^3\varphi_N \le w_N^3(-\varphi_N)\\
&\vdots\\
&v_N^n(-\varphi_N) \le (-1)^{n+1}V^n\varphi_N \le w_N^n(-\varphi_N), \quad n \in \mathbb{N}_+.
\end{aligned}$$

Corollary 3.3 *Let $f_N^0 \in C^1([0, 1])$ such that $(f_N^0)' > 0$. Put*

$$\alpha_N = \min_{x\in[0,1]} \frac{-\varphi_N(x)}{(f_N^0)'(x)} \text{ and } \beta_N = \max_{x\in[0,1]} \frac{-\varphi_N(x)}{(f_N^0)'(x)}.$$

Then

$$\frac{\alpha_N}{\beta_N} v_N^n (f_N^0)'(x) \le (-1)^n V^n (f_N^0)'(x) \le \frac{\beta_N}{\alpha_N} w_N^n (f_N^0)'(x), \quad n \in \mathbb{N}_+, \ x \in [0,1]. \tag{3.19}$$

Proof Noting that $\alpha_N (f_N^0)'(x) \le (-\varphi_N)(x) \le \beta_N (f_N^0)'(x)$ and using Remark 3.2 we can write

$$\begin{aligned}\frac{\alpha_N}{\beta_N} v_N^n (f_N^0)'(x) &\le \frac{1}{\beta_N} v_N^n (-\varphi_N)(x) \le \frac{1}{\beta_N} (-1)^{n+1} V^n \varphi_N(x) \\ &\le (-1)^n V^n (f_N^0)'(x) \le \frac{(-1)^{n+1}}{\alpha_N} V^n \varphi_N(x) \\ &\le \frac{w_N^n}{\alpha_N} (-\varphi_N)(x) \le \frac{\beta_N}{\alpha_N} w_N^n (f_N^0)'(x), \quad n \in \mathbb{N}_+,\end{aligned}$$

which shows that (3.19) holds. □

Theorem 3.4 *Let $f_N^0 \in C^1([0,1])$ such that $(f_N^0)' > 0$ and let μ be a probability measure on $\mathcal{B}_{[0,1]}$ such that $\mu \ll \lambda$. For any $n \in \mathbb{N}_+$ and $x \in [0,1]$ we have*

$$\begin{aligned}&\left(\log\left(\frac{N}{N-1}\right)\right)^2 \cdot \frac{N}{2} \cdot \frac{\alpha_N}{\beta_N} \min_{x\in[0,1]} (f_N^0)'(x) \cdot v_N^n G_N(x)(1-G_N(x)) \\ &\le |\mu(R_N^n < x) - G_N(x)| \\ &\le \left(\log\left(\frac{N}{N-1}\right)\right)^2 \cdot \frac{N}{2} \cdot \frac{\beta_N}{\alpha_N} \max_{x\in[0,1]} (f_N^0)'(x) \cdot w_N^n G_N(x)(1-G_N(x))\end{aligned} \tag{3.20}$$

where α_N, β_N, v_N and w_N are defined in Proposition 3.1 and Corollary 3.3, and

$$G_N(x) = \frac{1}{\log\left(\frac{N}{N-1}\right)} \log\left(\frac{x+N-1}{N-1}\right). \tag{3.21}$$

Proof For any $n \in \mathbb{N}$ and $x \in [0,1]$ set $d_n(G_N(x)) = \mu(R_N^n < x) - G_N(x)$, with G_N as in (3.21). Then by (3.1) we have

$$d_n(G_N(x)) = \int_0^x \frac{U^n f_N^0(u)}{u+N-1} \mathrm{d}u - G_N(x).$$

Differentiating twice with respect to x yields

$$d_n'(G_N(x)) \frac{1}{\log\left(\frac{N}{N-1}\right)} \frac{1}{x+N-1} = \frac{U^n f_N^0(x)}{x+N-1} - \frac{1}{\log\left(\frac{N}{N-1}\right)} \frac{1}{x+N-1},$$

$$(U^n f_N^0)'(x) = d_n''(G_N(x)) \left(\frac{1}{\log\left(\frac{N}{N-1}\right)}\right)^2 \frac{1}{x+N-1}, \quad n \in \mathbb{N}, x \in [0,1]. \tag{3.22}$$

Hence by (3.3) we have

$$\begin{aligned}d_n''(G_N(x)) &= \left(\log\left(\frac{N}{N-1}\right)\right)^2 (x+N-1) \left(U^n f_N^0\right)'(x) \\ &= (-1)^n \left(\log\left(\frac{N}{N-1}\right)\right)^2 (x+N-1) V_N^n (f_N^0)'(x),\end{aligned}$$

for any $n \in \mathbb{N}$, $x \in [0,1]$. Since $d_n(0) = d_n(1) = 0$, a well-known interpolation formula yields

$$d_n(x) = -\frac{x(1-x)}{2} d_n''(\xi), \quad n \in \mathbb{N}, x \in [0,1], \tag{3.23}$$

for a suitable $\xi = \xi(n,x) \in [0,1]$. Therefore

$$\begin{aligned}
\mu(R_N^n < x) - G_N(x) &= -\frac{G_N(x)(1-G_N(x))}{2} d_n''(G_N(\xi_N)) \\
&= (-1)^{n+1} \left(\log\left(\frac{N}{N-1} \right) \right)^2 (\xi_N + N - 1) V_N^n (f_N^0)'(\xi_N) \frac{G_N(x)(1-G_N(x))}{2} \\
&\le (-1)^{n+1} \left(\log\left(\frac{N}{N-1} \right) \right)^2 \frac{N}{2} V_N^n (f_N^0)'(\xi_N) G_N(x)(1-G_N(x))
\end{aligned}$$

for any $n \in \mathbb{N}$ and $x \in [0,1]$, and another suitable $\xi_N = \xi_N(n,x) \in [0,1]$. The result stated follows now from Corollary 3.3. □

4 Final remarks

We use the values obtained in "Appendix".

Let us consider the case $N = 3$. The equation $H_3(x) = 0$, with $e_3 = 0.8956735$, has as unique acceptable solution $t_3 = 0.4999967$. For this value of t_3 the function $\varphi_{t_3}/V\varphi_{t_3}$ attains its minimum equal to -4.76939599403913 at $x = 0$ and $x = 1$, and has a maximum $m(t_3) = (\varphi_{t_3}/V\varphi_{t_3})(0.423325998187593) = -4.62762782434937$. It follows that upper and lower bounds of the convergence rate are respectively $O(w_3^n)$ and $O(v_3^n)$ as $n \to \infty$, with $v_3 > 0.20967015556054$ and $w_3 < 0.216093436628214$.

Let us consider the case $N = 5$. The equation $H_5(x) = 0$, with $e_5 = 0.4088150$, has as unique acceptable solution $t_5 = 0.5000000$. For this value of t_5 the function $\varphi_{t_5}/V\varphi_{t_5}$ attains its minimum equal to -8.72035227508647 at $x = 0$ and $x = 1$, and has a maximum $m(t_5) = (\varphi_{t_5}/V\varphi_{t_5})(0.457198440687485) = -8.64358828233062$. It follows that upper and lower bounds of the convergence rate are respectively $O(w_5^n)$ and $O(v_5^n)$ as $n \to \infty$, with $v_5 > 0.114674266412028$ and $w_5 < 0.115692692356046$.

Finally, let us consider the case $N = 100$. The equation $H_{100}(x) = 0$, with $e_{100} = 0.0152027$, has as unique acceptable solution $t_{100} = 0.4999998$. For this value of t_N the function $\varphi_{t_{100}}/V\varphi_{t_{100}}$ attains its minimum equal to -198.668858764086 at $x = 0$ and $x = 1$, and has a maximum $m(t_{100}) = (\varphi_{t_{100}}/V\varphi_{t_{100}})(0.49804751660470764) = -198.66555309796482$. It follows that upper and lower bounds of the convergence rate are respectively $O(w_{100}^n)$ and $O(v_{100}^n)$ as $n \to \infty$, with $v_{100} > 0.00503350150708559$ and $w_{100} < 0.00503358526129032$.

$N = 3$	$v_3 > 0.20967015556054$	$w_3 < 0.216093436628214$
$N = 5$	$v_5 > 0.114674266412028$	$w_5 < 0.115692692356046$
$N = 100$	$v_{100} > 0.00503350150708559$	$w_{100} < 0.00503358526129032$

Appendix

Imposing conditions (3.18) and using MATHEMATICA we obtain

N	e_N	t_N
2	2.1780250	0.4999997
3	0.8956735	0.4999967
4	0.5616365	0.5000001
5	0.4088150	0.5000000
10	0.1730660	0.5000007
15	0.109754	0.4999972
20	0.080357	0.5000078
25	0.063380	0.4999999
30	0.052326	0.5000149
40	0.038793	0.4999972
50	0.030822	0.5000046
100	0.0152027	0.4999998
1000	0.00150201	0.5000073
10000	0.00015002	0.4999999

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