OVIDIUS UNIVERSITY OF CONSTANTA

DOCTORAL DOMAIN: MATHEMATICS

Metrical Theory of Some Continued Fraction Algorithms

ABSTRACT OF THE HABILITATION THESIS

Dr. Dan LASCU

2020

Abstract

This thesis represent the habilitation thesis of the author and contains the results achieved after 2010, the year when he obtained his PhD title in mathematics under the supervision of Acad. Marius Iosifescu.

The thesis is a systematic presentation of some continued fraction expansions that have been investigated by the author, individually or in collaboration, during the last 10 years.

To this day the Gauss map, on which metrical theory of regular continued fraction is based, has fascinated researchers from various branches of mathematics and science, with many applications in computer science, cosmology and chaos theory. In the last century, mathematicians broke new ground in this area. Apart from the regular continued fraction expansion, very many other continued fraction expansions were studied.

The metrical theory of the continued fraction expansion is about the sequence of its incomplete quotients, and related sequences and has as a starting point the studies of C.F. Gauss (1777 - 1855).

We should mention one problem formulated by Gauss and recorded in his diary on October 25, 1800, under the number 113. Twelve years later he wrote about this problem to Laplace in the letter dated January 30, 1812. Here's what Gauss wrote to Laplace (in an approximate translation):

"... I recollect a curious problem which I dealt with around 12 years ago but had no satisfactory solution to at the time. Should you permit yourself to spend several minutes solving it and I'm sure that you could find more complete solution. Here it is. Let M be an unknown value between 0 and 1 for which all its meanings are equally possible or adhere more or less to the same law. Assume that it can be represented by

continued fraction

$$M = \frac{1}{a^{(1)}} + \frac{1}{a^{(2)}} + \dots.$$

What is the probability that after exclusion of a finite number of terms up to $a^{(n)}$ one will have the remaining part

$$\frac{1}{a^{(n)}} + \frac{1}{a^{(n+1)}} + \dots$$

belonging to the interval from 0 to x? I denote this via $P(n, x)$ and assume that for M all its meanings are equally probable: $P(0, x) = x$."

Although he does not explicitly state it, Gauss must have known the recursion formula

$$P(n+1, x) = \sum_{i=1}^{\infty} \left(P\left(n, \frac{1}{i}\right) - P\left(n, \frac{1}{x+i}\right) \right),$$

since he wrote that he could prove by a "very simple reasoning" that (in modern notation)

$$\lim_{n\to\infty} P(n, x) = \frac{\log(1+x)}{\log 2}, \quad x \in [0, 1].$$

He wrote then that all his "attempts to solve the problem were in vain."

The thesis is structured in 5 chapters.

To make the thesis as self-contained as possible, we have included in the first chapter many of the notions that appear and repeat in the other chapters. Thus, we described the Banach spaces which are often mentioned throughout the thesis, we have mentioned some general concepts of ergodic theory, including Birkhoff Ergodic Theorem for measure preserving transformations, we gave some basic notions about the Markov chains and we explain Gauss' Problem and its progress. We have also given the definition and general properties of the Perron-Frobenius operator, an operator that plays an essential role in determining invariant measures and in studying their properties. Perhaps the most important part of the first chapter is its last section. Here we described the *random system with complete connections*, an exclusive product of the Probability School in Romania (M. Iosifescu [20]). We have given their most important properties, and also two examples.

In the 2nd chapter we collected all the results obtained by the author in the 3 articles published on the same subject [57, 37, 58]. Motivated by problems in random continued fraction expansions, S. Chakraborty and B.V. Rao [9] have initiated a systematic study of the continued fraction expansion of a number in terms of an irrational $\theta \in (0,1)$. This new expansion of positive reals is called *θ-expansion*. In the article in *Journal of Function Spaces* from 2014 [57] we made a detailed study of the metric theory of these expansions and proved the first Gauss–Kuzmin theorem for θ-expansions applying the method of random systems with complete connections. In the article in *Publicationes Mathematicae Debrecen* [37] from 2017, we used the Szüsz method [62] and we obtain a better rate of convergence than that obtained in the first article. Finally, in the article from 2019 published in *Journal of Number Theory* [58], we solved a Gauss–Kuzmin theorem related to the natural extension that generates the θ-expansion. Also, in the last section of the chapter we try to get close to the optimal convergence rate. Here, the characteristic properties of the Perron–Frobenius operator on the Banach space of functions of bounded variations allows us to derive explicit lower and upper bounds of the error term which provide a more refined estimate of the convergence rate involved.

In the 3th chapter we discussed about N-continued fraction expansions introduced by Burger et al. in 2008 [6]. We consider a family $\{T_N : N \in \mathbb{N}_+\}$ of interval maps as generalizations of the Gauss transformation. Then, by successive applications of the map T_N we obtain the N-continued fraction expansions of an irrational $0 < x < 1$. The chapter contains the results obtained in 3 articles [36, 35, 59] published between 2016-2020. Thus, in the article in *Mathematical Reports* [36] we discussed some metric properties of N-continued fraction expansions and we defined and investigated the associated Perron-Frobenius operator of T_N. We mention that the infinite-order-chain representation of the sequence of the partial quotients of the N-continued fraction expansion allows a concise formulation of the obtained results. Using the ergodic behavior of the random system with complete connections associated with N-continued fraction expansion, in the article in *Journal of Mathematical Analysis and Applications* [35] from 2016, we determined the limit of the sequence $\left(\mu\left(T_N^n < x\right)\right)_{n\in\mathbb{N}_+}$ of distributions, where μ is an arbitrary probability measure. In sections 3.5.1 and 3.5.2 we introduced the results from the article in *Publicationes Mathematicae Debrecen* [59] from 2020. Thus, in section 3.5.1 we presented a Gauss-Kuzmin theorem related to the natural extension of the measure-theoretical dynamical system associated to N-continued fraction expansions. Section 3.5.2 is devoted to deriving explicit lower and upper bounds of the error which provide a more refined estimate of the convergence rate. The key role in this section is played by the transition operator associated with the random system with complete connections underlying N-continued fraction on the Banach space of functions of bounded variation.

In the 4th chapter we collected the results contained in the 3 articles referring to *generalized Rényi continued fractions* [38, 39, 60]. In the article in *Acta Mathematica Hungarica* [38] from 2020, we started an approach to the metrical theory of Rényi-type continued fraction expansions via dependence with complete connections. More precisely, we obtained a version of Gauss-Kuzmin theorem for these expansions by applying the theory of random systems with complete connections. Once a finite ergodic measure ρ_N it is obtained for the map R_N which generates this expansion (recall that R_N is ergodic and measure preserving under ρ_N), classical results of ergodic theory, such as Birkhoff ergodic theorem, yield precise information on the frequency at which a digit occurs. It should be stressed that the ergodic theorem does not yield any information on the converge rate in the Gauss-Kuzmin problem that amounts to the asymptotic behavior of $\mu\left(R_N^{-n}\right)$ as $n \to \infty$, where μ is an arbitrary probability measure. So that a Gauss-Kuzmin theorem is needed. Using the natural extensions for the Rényi-type transformations, we obtained an infinite-order-chain representation $(\overline{a}_\ell)_{\ell\in\mathbb{Z}}$ of the sequence of the incomplete quotients $(a_n)_{n\in\mathbb{N}_+}$ of these expansions. Then we showed that the associated random systems with complete connections are with contraction and their transition operators are regular with respect to the Banach space of Lipschitz functions. This allowed us to solve a variant of the Gauss-Kuzmin problem. In this result the constants involved are far from optimal. So a hunt for the best possible constants started. In the article in *Acta Arithmetica* [39] from 2020 we continue our investigation on the asymptotic behavior of the distribution functions of the Rényi-type transformations. In order to prove a Gauss-Kuzmin-Lévy-type theorem for the Rényi-type continued fraction expansions, we apply the method of Szüsz [62]. We mention that using this method, we obtain more information on the convergence rate involved for which we give an explicit expression in terms of Hurwitz zeta functions. But even this is not the optimal convergence rate. For this reason, in the article published in *Periodica Mathematica Hungarica* [60] we used a Wirsing-type approach to get close

to the optimal convergence rate. By restricting the domain of the Perron-Frobenius operator of R_N under its invariant measure ρ_N to the Banach space of functions which have a continuous derivative on $[0, 1]$, we obtain upper and lower bounds of the error which provide a refined estimate of the convergence rate.

In the last chapter we briefly present further developments of author's academic and scientific career. The thesis ends by displaying 64 references in the Bibliography, all cited in the text.

Acknowledgments

A special gratitude goes to my PhD supervisor, Acad. Marius Iosifescu, from "Gh. Mihoc-Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, who influenced my carrier from the very beginning.

I hereby want to thank to my collaborator Dr. Gabriela Ileana Sebe from Politehnica University of Bucharest, with whom I have worked excellently in the almost 10 years since we know.

Also, special thanks to my family, especially to my wife and our daughter, who allowed me to postpone certain household activities to give me time to study.

Constanta, May 2020 D.L.